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REVIEWS

Measurement and Physical Content of Quantum Information

B. A. Grishanin and V. N. Zadkov

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Abstract—Basic quantum information measures involved in the information analysis of quantum systems are considered. It is shown that the main quantum information measurement methods depend on whether the corresponding quantum events are compatible or incompatible. For purely quantum channels, the coherent and compatible information measures, which are qualitatively different, can be distinguished. A general information scheme is proposed for a quantum-physical experiment. In this scheme, informational optimization of an experimental setup is formulated as a mathematical problem.

INTRODUCTION

Significant progress in quantum physics in the last decade has essentially influenced the evaluation of the role and qualitative content of quantum physics, though, certainly, its fundamentals have been preserved. In the earlier studies of specific nonclassical features of quantum systems, an experimenter only had to ensure suitable macroscopic conditions (via choosing an object, using macroscopic fields, providing for necessary temperature, etc.). But today, it has become possible to intentionally vary directly quantum states of elementary quantum systems. This possibility initiated a number of novel applied sciences and technologies, such as quantum cryptography, quantum communications, and quantum calculation physics [1–4], which exploit nonclassical features of quantum system states. A comprehensive analysis of quantum specific features characterizing physical systems (these features are involved in the above applications) can be found in the latest reviews [5–10] and monographs [11–14]. Despite the variety of physical mechanisms used for generation, processing, and transmission of quantum information, all of them are principally based on the only essential difference between quantum and classical events. This is noncommutativity of quantum variables of systems under consideration, which is equivalent to the nonorthogonality of their quantum states. Because of this circumstance, it is impossible to consider an arbitrary set of quantum events within the framework of classical logic, which is the result of the so-called incompatibility of nonorthogonal states.

Indeed, suppose two quantum states $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal. Then, using them as an algebraic basis for an algebra with addition in the form of linear subspace union (sp) and product in the form of intersection, we find that the algebra of quantum events constructed on $|\alpha\rangle$ and $|\beta\rangle$ contains only four subspaces $\{\varnothing, |\alpha\rangle, |\beta\rangle, H\}$ including the empty ($\varnothing$) and 2D Hilbert ($H = \mathrm{sp}(|\alpha\rangle, |\beta\rangle)$) spaces. It is equivalent to the algebra of four elements $\{\varnothing, \alpha, \beta, \mathcal{M}\}$, $\mathcal{M} = \alpha \cup \beta$ constructed as an aggregate of subsets contained in the set of two point elements α and β (the indices of quantum states under consideration) with addition of subsets in the form of their sum and product in the form of their intersection. This algebra, considered as an elementary example, corresponds to the classical (bivalent, i.e., Aristotelian [15]) logic underlying classical physics, where the rule of the excluded third holds: either *a* or not *–a*. In the above example, this rule is expressed as

$$\alpha \cup \beta = \mathcal{M}$$

in terms of indices and as

$$\mathrm{sp}(|\alpha\rangle, |\beta\rangle) = H$$

in terms of states. Here, $|\beta\rangle$ has the sense of negation of event $|\alpha\rangle$ and H has the sense of a known certain event. Note that both elementary events $|\alpha\rangle$ and $|\beta\rangle$ belong to this class of events. If two states are nonorthogonal, in the relationship $\mathrm{sp}(|\alpha\rangle$, state $|\beta\rangle) = H$ is not negation of $|\alpha\rangle$ because it includes a nonzero projection on $|\alpha\rangle$ along with the infinite set of other states $|\gamma\rangle$ existing due to the superposition principle. In this case, $|\alpha\rangle$ and $|\beta\rangle$ are, respectively, eigenvectors of two noncommuting operators $\hat{A}$ and $\hat{B}$ of certain physical quantities. These quantities cannot be measured simultaneously since their quantum eigenstates are nonorthogonal. When the above physical quantities take their possible values, the corresponding sets of quantum states are incompatible because they cannot be considered simultaneously within the framework of classical logic.

Below, we present an elementary analysis which reveals a close relationship between physical quantities and their corresponding quantum states. Determination of the relationships directly between coupling states and the quantitative measurement of information carried by these states is the subject of the information theory. The purpose of this study is to analyze the choice of an adequate quantitative measure of quantum information and its possible role in physics. To this end, we investigate a number of basic physical models of quantum systems and schemes of physical experiments in the presence or absence of incompatibility of quantum states, which is their only fundamental specific feature causing various quantum effects. In Section 1, the main types of quantum information and corresponding quantum measures are classified using this criterion. In Sections 2 and 3, we successively consider coherent and compatible information. The latter type of quantum information is closely related to quantifying information efficiency of a given experimental scheme. This problem is discussed in Section 4.

1. INCOMPATIBILITY OF QUANTUM EVENTS AS THE BASIS OF NONCLASSICAL SPECIFICITY AND CLASSIFICATION OF QUANTUM INFORMATION

The notion of quantum information formulated simultaneously with the basic laws of quantum physics is directly related to them and plays a key role in their interpretation. Any quantum effect (for example, the essentially microscopic process of the atom spontaneous radiation or a macroscopic transition to the superconducting state) can explicitly be related with the processes of quantum information transformation if this information is adequately associated with the corresponding ensembles of quantum states. One can also state that a prototype of quantum information had appeared before the classical Shannon information theory was developed. It suffices to recall the Born interpretation of the physical sense of the wave function or to analyze the information content of the quantum measurement postulate (the wave function collapse) [16].

The notion of quantum information was admitted to be important long ago. However, interest in its application has only recently been stimulated by the development of modern quantum optics experimental methods ensuring quantum system control. Progress in this area of quantum physics enables one to employ quantum information not only as a useful abstract notion but, also, to manipulate it in actual experiments in a free manner. Studies of quantum information processing initiated quantum information physics. This new field of physics is covered in the literature referred to in the bibliography index [17].

Proceeding from the conventional description of quantum mechanics [16, 18–20], one can suppose that proper physics must deal only with quantum-physical quantities, whereas quantum states could rather be studied, irrespective of specific physical variables, within the framework of mathematics, one of whose fields is sometimes qualified as the classical information theory. However, a more comprehensive analysis shows that this is not true: as soon as quantum states are associated with eigenstates of physical variables characterizing an actual quantum model, they become carriers of physically meaningful information. For example, let us consider the mathematical structure of self-adjoint operators $\hat{A}$ in Hilbert space H, which are used in quantum-mechanical representations of physical variables. Then, the spectral decomposition of operator $\hat{A} = \sum \lambda_n |n\rangle\langle n|$ describes its splitting into mathematical objects of two types: the set of physically possible values λ_n and the set of corresponding quantum states $|n\rangle$. The latter objects contain the most general physical information which is independent of specific values λ_n and characterizes only certain physical events. Each of these events lies in the fact that a physical quantity takes one of the values λ_n.

Information relationships between quantum states are determined by the dynamic properties of a physical system and, evidently, provide for the most fundamental description of its dynamic characteristics. These relationships may characterize the intrinsic dynamics of a quantum system and its interaction with other systems. Initially, they are represented as equations for wave functions or quantum state operators. The theoretical information approach is based on introducing an adequate quantitative measure of information exchange. In the general case, being independent of specific physical variables, this measure is superior to them in the analysis of general dynamic properties of a quantum system.

If a specific scheme of a physical experiment is not discussed, a quantum system is described using its information characteristics which yield quantitative relationships between the quantum states of a given system and the quantum states of other systems that may interact with it. For example, the main information content of a two-level atom radiation is reduced to the fact that the quantum information carried by this atom is transferred to the corresponding quantum of the photon field. In this situation, the quantum receives information on the phase of the initial atom state, i.e., the information exchange is essentially quantum and retains coherence of the transformed wave functions (see Section 2). Thus, the information description is more comprehensive than in the case when it is represented in terms of the atom–field energy exchange.

The fundamental concept of classical information measure is formulated in the Shannon information theory [21, 22]. For classical systems of physical events, a unified quantitative measure of information can be introduced, which is independent of neither the specific physical content of this information nor its usage. This measure enables one to express the asymptotic level of error-free transmitted information content as the optimized Shannon information content, which is necessary for combining numerous channels in the presence of noise.

This elegant theory is based on a special property of classical ensembles that is excluded from the original principles of quantum physics. This property is reproducibility of classical events: statistically, it makes no difference whether one and the same physical system or its information-equivalent copy is at the input and output. However, the latter situation is impossible in the quantum world. Obviously, this circumstance initiated the discussion of the question whether the Shannon information measure could be applied to quantum systems [23–25].

In this paper, it is shown that the traditional definition of the Shannon entropy and the corresponding information measure can successfully be used for analyzing quantum systems also in the case when fundamental differences between ensembles of classical and quantum events are properly taken into account.

The quantum theory involves the principle of quantum state superposition implying the existence of an arbitrary linear combination $c_1|\alpha\rangle + c_2|\beta\rangle$ of states $|\alpha\rangle$ and $|\beta\rangle$, which results in the presence of a continuum set in any quantum system. This set is Hilbert space $H \ni \psi$ of states, most of which do not coincide with any orthogonal basis states $|n\rangle$ associated with a certain physical quantity described by the operator $\hat{A} = \sum \lambda_n |n\rangle\langle n|$. When a quantum system is transformed, the basis vectors and the entire Hilbert space are also transformed. This circumstance, which is exploited in algorithms of quantum calculations, significantly enhances their efficiency due to high concurrency of operations [3, 4, 13]. However, the above continuum set does not contain an unlimited information content in the conventional classical sense.

The point is that an arbitrary quantum state $|\alpha\rangle \in H$ can be distinguished from another state $|\beta\rangle$ only when the states are orthogonal. The probability of random coincidence of two arbitrary states is determined by the squared absolute value of the scalar product, so that the 2D probability density of two equiprobable states has the form

$$P(d\alpha, d\beta) = |\langle\alpha|\beta\rangle|^2 \frac{dV_\alpha dV_\beta}{D}, \tag{1}$$

where dV_α and dV_β are small volume elements on the sphere of wave functions satisfying the normalization $\int |\alpha\rangle\langle\alpha| dV_\alpha = \hat{I}$ and D is the dimension of space H. Within the framework of the Shannon information theory, one can find effective number N_α of states of α distinguishable by variable β and vice versa [26]. Probability distribution (1) describes the information exchange between two information variables α and β, which, according to the Shannon theory, is characterized by effective number N_α of error-free transmitted messages that are formed of subsets A_α of indices corresponding to quantum states α. This effective number (calculated per symbol) is attained for infinitely long sequences of independently transmitted unit symbols. When each A_α is associated with appropriate states α, all of these states will be distinguished without error (in the above sense), so that N_α is the number of distinguishable states. It is determined by the corresponding Shannon information content

$$I_{\alpha\beta} = \int \log_2 \frac{P(d\alpha, d\beta)}{P(d\alpha)P(d\beta)} P(d\alpha, d\beta)$$

using the formula $N_\alpha = 2^{I_{\alpha\beta}}$. For joint probability distribution (1) (see [27]), $I_{\alpha\beta} = 1 - 1/\ln 4 \simeq 0.27865$ bit for $D = 2$ and $I_{\alpha\beta} = (1 - \mathbf{C})/\ln 2 \simeq 0.60995$ bit ($\mathbf{C}$ is the Euler constant) for $D \longrightarrow \infty$. Thus, at any space dimension D, $N_\alpha < 2$, i.e., quantum uncertainty reduces the effective number of distinguishable states even below a value of two corresponding to one bit. This is caused by high quantum uncertainty in pure states ψ, which, on the one hand, are analogs of deterministic classical states when only their orthogonal sets are considered and, on the other hand, contain internal quantum uncertainty. In the latter case, other nonorthogonal states may exist and pure states $P(x) = |\psi(x)|^2$ determine the corresponding probability distributions $P(x)$ for variables $\hat{x}$ such that ψ is not their eigenfunction. Particularly, the corresponding entropy of the N-partite state $\Psi_N = \psi \otimes \ldots \otimes \psi$ asymptotically approaches $N\log_2 D$ at $N \longrightarrow \infty$, i.e., the pure character of the state virtually does not influence the uncertainty of all of the quantum states. The mixed state $\hat{I} \otimes \ldots \otimes \hat{I}/D^N$ exhibits the maximum uncertainty equal to $N\log_2 D$ bit.

Thus, the first specific feature of quantum information due to incompatibility of quantum states is the impossibility of extracting a noticeable information content with spaces of large dimension D without selecting states. Quantum information always must be selected in transmission channels transforming a considerable information content in a distinguishable form.

Let us consider a simple example illustrating the qualitative characteristics of quantum systems due to incompatibility of all quantum states. Suppose two two-level atoms are in one and the same state (Fig. 1a). While the operational sense of the expression *in one and the same state* is quite clear, its qualitative sense essentially does not coincide with that of this expression in the classical case. When this example is considered from the classical standpoint, only two basis states ($k = 1, 2$) of each atom are taken into account. Then, when describing the statistic of these atom states, it makes no difference whether they are assumed to correspond to different atoms or to one and the same atom. The point is that, in a combined system, only one state has a nonzero probability and the knowledge of state k of each atom corresponds to the exact knowledge of a possible state of the other atom. Thus, when only populations are considered in the classical case, atoms can be equivalent copies of each other.

In the quantum case, it is impossible to copy all quantum states in a similar manner. In addition to the above two states in an atom, there exist other states $|\alpha\rangle$

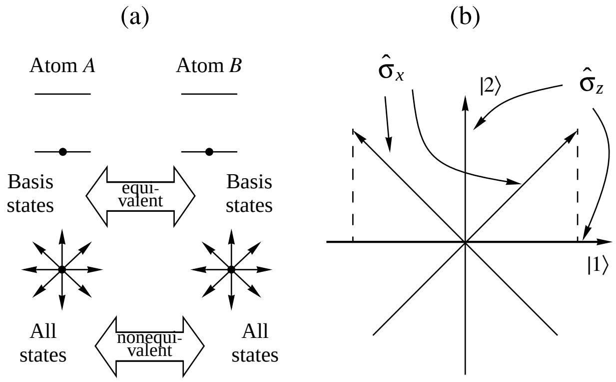


Fig. 1. Incompatibility of nonorthogonal quantum states. (a) Equivalence of compatible ensembles of basis states and nonequivalence of complete quantum state ensembles for two two-level atoms. (b) Vacuum fluctuations caused by incompatibility.

with nonzero probabilities $|\langle\alpha|k\rangle|^2$ such that $|k\rangle$ is not the proper basis for averaged physical quantities. This circumstance is due to quantum uncertainty, which is always present in an ensemble of quantum states (Fig. 1b). It is well known that, for a harmonic oscillator, this uncertainty manifests itself in nonzero energy of vacuum fluctuations $\hbar\omega/2$. For two-level atoms, it takes the form of nonzero values $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{I}$, where Pauli matrices $\hat{\sigma}_{x,y}$ have the sense of quadrature cosine and sine components of atom oscillators. In spite of the fact that these atoms are in one and the same state, their corresponding eigenstates are different because the above state refers to different physical systems. Each of these systems contains its own ensemble of mutually incompatible quantum states, which are described by nonorthogonal eigenvectors corresponding to different noncommuting operators of physical variables as shown in Fig. 1b. Indeed, mean-square residuals $(\hat{\sigma}_x^A - \hat{\sigma}_x^B)^2$ and $(\hat{\sigma}_y^A - \hat{\sigma}_y^B)^2$ are nonzero because their operators do not commute with the operator of difference of populations $\hat{\sigma}_z$ which corresponds to strictly nonzero difference $\hat{\sigma}_z^A - \hat{\sigma}_z^B$. This means that the eigenstates of these quantum variables for the atoms under consideration do not coincide, i.e., all of their mutually corresponding quantum states are by no means copies of each other.

This simple example implies the important general conclusion that, in the presence of incompatibility, i.e., nonorthogonality in ensembles of quantum states of two different atoms, these ensembles are always different. Hence, information on the states of a quantum system at a certain instant which is obtained by means of any other system considered at the same instant is never complete. Complete quantum information on all quantum states of a system at a specified instant can be provided only by this system itself. The complete information content can appear in another place (or at another instant) when it is automatically canceled at the initial position, which occurs, for example, during teleportation [2]. Quantum information can be teleported only to a single receiver, which provides for the possibility of designing absolutely intercept-secure communication systems based on quantum cryptography.

The above qualitative concept of uniqueness of quantum information can be justified quantitatively. Let us consider the squared difference of projectors on the mutually corresponding wave functions of two atoms considered at one and the same instant. Integrating the operator of this squared difference over all possible wave functions, we obtain the expression yielding a strictly positive value:

$$\hat{\varepsilon} = \int (|\alpha\rangle\langle\alpha| \otimes \hat{I}_B - \hat{I}_A \otimes |\alpha\rangle\langle\alpha|)^2 \frac{dV_\alpha}{D}$$
$$= \|0\rangle\rangle\langle\langle 0\| + \frac{1}{3}\sum_{k=1}^{3} \|k\rangle\rangle\langle\langle k\| \geqq \frac{1}{3}, \tag{2}$$

where integration is performed over the Bloch sphere including states $|\alpha\rangle$ with index $\alpha = (\varphi, \vartheta)$, $dV_\alpha = \sin\vartheta d\vartheta d\varphi/(2\pi)$ is a small volume element, and $V_\alpha = D = 2$ is the total volume of integration. This expression is similar to the corresponding classical formula for the rms discrepancy

$$\varepsilon = \sum_{\xi} (\delta_{\xi_A\xi} - \delta_{\xi_B\xi})^2$$

between classical indicators $\delta_{\xi_A\xi}$ and $\delta_{\xi_A\xi}$ of events $\xi_A = \xi$ and $\xi_B = \xi$ related with random variables ξ_A and ξ_B. For any joint probability distribution $P(\xi_A, \xi_B) = P(\xi_A)\delta_{\xi_A\xi_B}$ that describes random variables coinciding everywhere, the mean of random function ε is zero, i.e., for these probability distributions, all of the events $\xi_A = \xi$ and $\xi_B = \xi$ are realized simultaneously and the above quantities are copies of each other. Being an exact quantum analog of the discrepancy between two ensembles of classical events, bipartite operator (2) has two proper subspaces which are formed of singlet and triplet Bell states $\|k\rangle\rangle$ and correspond to the eigenvalues $\varepsilon_k = 1, 1/3$ of the mean-square residual. The value corresponding to the singlet state is three times greater than the value corresponding to the triply degenerate triplet state. Strict positivity of this operator, i.e., the absence of the zero eigenvalue, means that, for any joint density matrix, its averaging yields a nonzero result, which determines the rms discrepancy of all quantum states. Hence, it is impossible to mutually copy all quantum states of various systems irrespective of their states.

The foregoing implies that the key difference between the classical and quantum types of information depends on whether the states associated with the information of interest are compatible or incompatible. The states of different systems considered at one and the same instant are always compatible. Hence, in the presence of internally incompatible states, they cannot copy each other, and the total ensembles of quantum states of one and the same system which are considered at two different instants are most often incompatible and, moreover, can copy each other in the absence of noise retaining the uniqueness of their quantum fluctuations at every instant. However, the states of two different systems observed at different instants may be compatible or incompatible depending on the transformation which couples these two instants. This factor is rather important for the basic definitions of quantitative quantum information measures (see Sections 2 and 3).

Thus, according to the most fundamental classification of quantum information, which involves the compatibility/incompatibility property of state ensembles under consideration, the following four types of information can be recognized: classical information, semiclassical information, coherent information, and compatible information.

In the case of classical information, all states are compatible and, within the original form of the Shannon information theory, are considered as classical on default [21, 22]. However, classical information can always be transmitted via a quantum channel and is of interest to quantum physicists as well. A classical channel is specified by conditional probability distribution $p(y|x)$ of states of output y at fixed states of input x.

For semiclassical information, all input information is specified by classical states λ and output states are characterized by internal quantum incompatibility as quantum states in Hilbert space H. Nevertheless, output states are automatically compatible with input states. In the general case, a quantum channel is described by an ensemble of mixed quantum states $\hat{\rho}_\lambda$ depending on classical parameter λ [28, 29]. Variables λ are equivalent to input variables x, the set of all wave functions $\psi \in H$ is equivalent to output states y, and the density matrix of $\hat{\rho}_\lambda$ is equivalent to conditional probability distribution $p(y|x)$ of a classical channel.

In the case of coherent information, the spaces of input and output states are characterized by internal quantum incompatibility. Being related by channel superoperator $\mathcal{N}$, which transforms the input density matrix to the output density matrix, $\hat{\rho}_B = \mathcal{N}\hat{\rho}_A$ [30, 31], these spaces are mutually incompatible. Transformation $\mathcal{N}$ determines the flow of quantum incompatible states transmitted from the channel input to output and is a completely quantum analog of classical conditional distribution $p(y|x)$, which, in a similar manner, linearly transforms classical input probability distribution $p(x)$ to output distribution $p(y)$.

In the case of compatible information, the input and output contain internally incompatible states but are mutually compatible.

While the first three types of information and the coherent information measure (which was introduced not long ago) are well known [22, 28, 30], compatible information has been introduced in an explicit form most recently as a special type of information measure [32]. It is determined for a combined bipartite system with the compatible input and output characterized by internal quantum incompatibility.

Coherent and compatible types of information comprise all possible qualitatively different types of information in completely quantum channels. Our study of possibilities provided by applying the information approach to actual experiments shows that only compatible information is an adequate tool for analyzing the information efficiency of an abstract scheme of a quantum-physical experiment.

2. COHERENT INFORMATION

A. The Physical Sense of Coherent Information

According to the classification presented in Section 1, one possible quantitative information measure for a completely quantum channel is coherent information introduced by Schumacher and Lloyd [30, 31]. It serves as a quantitative measure of incompatible information content that is transmitted from one state space to another. One can consider both the case of one and the same state space and the case of physically different spaces. A trivial case of coherent information exchange is the dynamic evolution of a closed system described by unitary operator U: $\hat{\rho}_B = U\hat{\rho}_A U^{-1}$. Here, all pure states ψ admissible by initial density matrix $\hat{\rho}_A$ are transmitted unchanged and transmitted coherent information content I_c coincides with the initial information content. By definition, the latter is measured with the Von Neumann entropy $S[\hat{\rho}_B] = S[\hat{\rho}_A]$, i.e., the corresponding information is described by the expression

$$I_c = -\mathrm{Tr}\hat{\rho}_A \log\hat{\rho}_A. \qquad (3)$$

However, this definition requires additional reasoning in terms of the operational sense of the density matrix. In the self-consistent theory, this matrix is only a result of averaging the pure state of a combined system over auxiliary variables. Then, expression (3) must be considered as the entanglement of input system A and reference system R for properly chosen pure state Ψ_{AR} (such that $\mathrm{Tr}_R|\Psi_{AR}\rangle\langle\Psi_{AR}| = \hat{\rho}_A$) of combined system $A + R$. Thus, coherent information is measured in terms of mutually compatible states of two different systems A and R. At the same time, information is transferred from input A to output B, the latter differing from A by a unitary transformation only.

Information channel $\mathcal{N}$ with corresponding noisy environment E (Fig. 2a) must be included in the information system, which completes the description of its general structure [33].

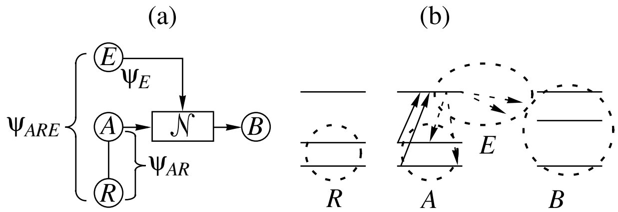


Fig. 2. (a) Block diagram of a quantum information system and (b) its possible physical realization: (A) input, (B) output, (R) reference system, and ($\mathcal{N}$) channel with noisy environment E.

Coherent information for a channel of the general type is defined as follows [33]:

$$I_c = S[\hat{\rho}_B] - S[(\mathcal{N} \otimes \mathcal{I})|\Psi_{AR}\rangle\langle\Psi_{AR}|], \tag{4}$$

which is in agreement with the Shannon definition of classical information [34]. Here, $\mathcal{I}$ is the unit superoperator which is applied to the variables of the reference system leaving them unchanged. The second term in this expression describes the exchange entropy, which is nonzero only due to the interaction between subsystems $A + R$ and E when $\mathcal{N} \neq \mathcal{I}$. Superoperator $\mathcal{N}$ transforms, according to the relationships

$$\hat{\rho}_B = \mathcal{N}\hat{\rho}_A = \mathrm{Tr}_R\hat{\rho}_{BR}, \quad \hat{\rho}_{BR} = (\mathcal{N} \otimes \mathcal{I})|\Psi_{AR}\rangle\langle\Psi_{AR}| \tag{5}$$

the states of input A into the states of output B, whose quantum states are compatible with the states of reference system R as before because these systems are not entangled due to the above transformation. This enables one to consider B and R as kinematically independent systems described by joint density matrix $\hat{\rho}_{BR}$. Taking into account the above circumstance and the zero, by supposition, entropy of combined system $R + B + E$, relationship (4) can be considered as the measure of entanglement between output B and combined system $R + E$ decreased by the entropy of exchange between the channel $A \longrightarrow B$ and noisy environment E. Thus, it follows from (4) that, in terms of physical content, coherent information is a specific measure of preserved entanglement between compatible systems R and A (remaining after information is transmitted through the channel $A \longrightarrow B$) rather than that it directly serves as a measure of quantum incompatible state flow transmitted from A to B. In the general case, output B may be physically different from A and even may be described by a Hilbert space of quite different dimension ($H_B \neq H_A$) [35, 36], which is illustrated by the physical example of the information system presented in Fig. 2b.

For this system, input A and reference system R correspond to the ground two-level states of two entangled atom Λ systems. Information channel $\mathcal{N}$ is provided by laser excitation of input system A at the radiation-active upper level. Combined with the vacuum state, two emitted photons correspond to output B, whereas all of the remaining freedom degrees of the field combined with the excited atom state form noisy environment E.

The elementary carrier of quantum information is a two-level system, an analog of the classical bit which is conventionally called a qubit.[1] Therefore, it is logical to call the quantitative measure of specifically quantum coherent information a qubit too. Obviously, this unit of coherent information corresponds to the use of a binary logarithm in definition (3), which yields $I_c = 1$ qubit for a two-level system with density matrix $\hat{I}/2$ characterizing the state with the maximum possible quantum entropy. With this state, all possible quantum states are presented equally and most completely.

Now, let us find out how the quantitative measure of quantum information can be used in physics. The quantum theory usually is applied to calculations of certain means of the form $\langle\hat{A}\rangle = \sum\lambda_n\langle|n\rangle\langle n|\rangle$, where λ_n and $|n\rangle$ are, respectively, eigenvalues and eigenvectors of operator $\hat{A}$. This decomposition is the result of averaging physical quantities represented in terms of probabilities $P_n^A = \langle|n\rangle\langle n|\rangle$ of quantum states $|n\rangle$. All possible variables constitute an infinite set that is much richer than the set of all quantum states. Therefore, irrespective of physical values, relationships between physical states provide for more general information on physical couplings in a more compact form. The characteristic features of the coherent information exchange correspond to physical relationships expressed in the most general form because they are associated with the most general properties of interaction between two quantum systems chosen as an input and output and related by a one-to-one transformation of all possible mixed input states. Actually, the dependence of coherent information on the parameters of an information system is of even more fundamental character than relationships between specific physical quantities. This dependence is calculated for a number of fundamental models widely applied in quantum physics [34–36].

As an example, let us consider the well-known Dicke problem [37], where the information exchange between atoms demonstrates the dynamics of the same oscillation type [35] as the energy exchange via emitted photons, which determines radiation damping in a system of two two-level atoms. In this case, the oscillation dynamics is typical of not only energy but also the set of other variables. Hence, in order to describe the general properties of interatomic interaction, it is expedient to consider coherent information as a measure of preserved entanglement rather than the set of various physical variables. The former is a characteristic of the internal quantum incompatibility exchange between mutually compatible state sets of the reference and input systems H_R and H_A. In the Dicke problem, information exhibits its coherent character in the time dependence

[1] The term *qubit* was first introduced by B. Schumacher [51].

at a large difference between decay rates of the short- and long-lived Dicke states. The interatomic exchange coherence is realized by coherent oscillations between both of these components. Therefore, the lifetime of coherent information is determined by the short-lived component in contrast to, for example, the lifetime of the atom population determined by the long-lived state.

Unlike the other types of quantum information, coherent information enables one to distinguish two qualitatively different information exchange classes corresponding to the cases when classical information or quantum state entanglement is used. Only in the latter case, coherent information is nonzero. Therefore, it is just coherent information which can be adequately used for finding out to what extent a quantum information transmission channel retains the capability of utilizing the output as an input equivalent for realizing situations where the quantum character of an input signal is of essential importance. These problems are widely covered in the modern literature (see [13] and references cited therein). Being a measure of the entanglement of a quantum system that is preserved by a physical transformation, coherent information is now of certain practical interest for quantum information transmission and processing as well as for the analysis of specific physical models of quantum channels illustrated below by an example.

B. One-Time Coherent Information

Proceeding from formal mathematical analogues, we can start describing a two-sided quantum information channel with the formal quantum generalization of the Shannon classical mutual information $I = S_A + S_B - S_{AB}$:

$$I = S[\hat{\rho}_A] + S[\hat{\rho}_B] - S[\hat{\rho}_{AB}]. \qquad (6)$$

This generalization makes sense only when joint density matrix $\hat{\rho}_{AB}$ is specified. This matrix is considered as a direct analog of classical joint probability distribution P_{AB} [38].

Obviously, formula (6) can be applied to quantum systems if we assume that the states of systems A and B are mutually compatible. It is *a fortiori* true for one-time states of the corresponding physical systems when they are not overlapped by the parts of one and the same quantum system containing both an input and output.[2] The foregoing justifies the term *one-time* as referred to information that is specified using the joint density matrix as an initial characteristic. Though one can formally define quantum information I with expression (6), its physical sense remains to be seen [40, 41]. This can be attributed to the main qualitative difference between the classical and quantum information channels. In general, relationship (5) implies that the quantum input and output are incompatible as, for example, in the case when the states of one and the same quantum system are considered at two different instants. Thus, A and B in (6) cannot be the input and output that are involved in the definition of coherent information. Therefore, in the quantum case, in order to apply density matrix $\hat{\rho}_{AB}$, one has to physically specify systems A and B. This can be done using the Schumacher definition of coherent information, which inevitably necessitates modifying expression (6).

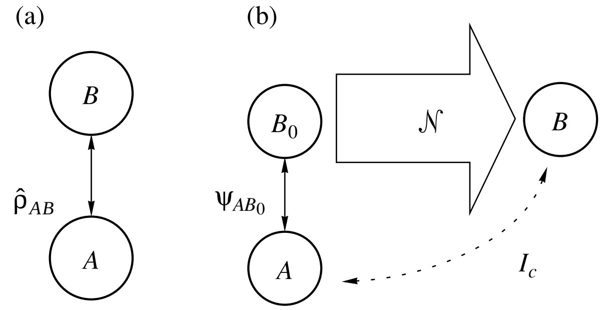


Fig. 3. Reconstruction of a quantum information system corresponding to given joint density matrix $\hat{\rho}_{AB}$: (a) mathematical description of the channel specifying one-time information and (b) its correspondence with the Schumacher construction [33].

Adequate physical interpretations of systems A and B are provided by identifying them, respectively, with the reference system and with the output of a certain quantum channel associated with specified joint density matrix $\hat{\rho}_{AB}$ as shown in Fig. 3. In this case, they are automatically compatible. Therefore, the actual input of the above channel corresponds to a certain state of input B_0 at an initial instant rather than to system A. At the initial instant, we have information on A which is not distorted by the channel and is transformed by $\mathcal{N}$ into the final state of output B. In this case, B_0 and channel $\mathcal{N}$ manifest themselves in the form of density matrix $\hat{\rho}_{AB}$ coupling the output and reference system rather than being explicitly introduced. Specified density matrix $\hat{\rho}_{AB}$ properly corresponds to its analog, i.e., density matrix $\hat{\rho}_{BR}$ in (5) (which is involved in the Schumacher method), when the structure consisting of the reference system and input is characterized by a pure state described by function Ψ_{AB_0} such that the relationship

$$\hat{\rho}_{AB} = (\mathcal{I} \otimes \mathcal{N})|\Psi_{AB_0}\rangle\langle\Psi_{AB_0}| \qquad (7)$$

holds for a certain channel $\mathcal{N}$. Then, the density matrix

$$\hat{\rho}_A = \mathrm{Tr}_{B_0}|\Psi_{AB_0}\rangle\langle\Psi_{AB_0}|$$

[2] A similar generalized meaning of coherent information and its calculation for specific systems are also possible [35]. When systems move at relativistic velocities, the appropriate relativistic corrections are necessary. These situations are topical in modern experiments with entangled quantum states [39] where effects associated with motion of a measuring system are recorded.

of the reference system can automatically be represented as the corresponding partite density matrix $\mathrm{Tr}_B\hat{\rho}_{AB}$ of the system $A + B$ if one takes into account that the trace with respect to B_0 in (7) is $\mathcal{N}$ invariant because this transformation does not affect subsystem A.

The foregoing implies that the corresponding one-time coherent information can be defined as the Schumacher coherent information

$$I_c = S[\hat{\rho}_B] - S[\hat{\rho}_{AB}], \tag{8}$$

which, unlike information measure (6), does not contain the term $S[\hat{\rho}_A]$. As follows from the above description, in the general case, the term *one-time* does not directly mean that it refers to the coupling between the system states at one and the same instant. Actually, reference system A and output B may be considered at different instants, and the only important condition is their compatibility. Thus, one-time information couples two compatible systems all of whose variables are mutually compatible, i.e., described by commuting operators, in contrast to coherent information, which, in the general case, couples the incompatible input and output.

Representing coherent information in another form, one-time information (8) (in contrast to information (6)) is not symmetrical with respect to A and B. Moreover, coherent information may be negative. The latter circumstance is evident for density matrices $\hat{\rho}_{AB}$ corresponding to a purely classical exchange between basis sets of orthogonal states: $\hat{\rho}_{AB} = \sum P_{ij}|i\rangle\langle j||j\rangle\langle i|$. Then, all entropies are reduced to classical ones: $S[\hat{\rho}_{AB}] = S_{AB} = -\sum P_{ij}\log P_{ij}$, $S[\hat{\rho}_B] = S_B = -\sum P_j \log P_j$, and $S_{AB} > S_B$. A negative value of coherent information means that the exchange entropy is so large that it not only makes the preserved information content vanish but also exceeds the corresponding critical value. In the latter case, one can assume that $I_c = 0$.

C. The Rate of Coherent Information Exchange in Λ Systems

The information system presented in Fig. 2b plays a special role in modern applications based on nonclassical features of quantum information, such as quantum cryptography and quantum computations. Atomic Λ systems can be used as basic blocks in these applications. They are promising carriers of elementary quantum information units (qubits) that enable one to efficiently store quantum information and freely manipulate it by means of laser radiation [10, 13]. For the information system presented in Fig. 2b, the use of the second Λ system as a reference system is physically justified because the entanglement of two corresponding qubits has a clear physical meaning as initially stored quantum information. Particularly, the latter can be applied for performing basic logical operations in quantum computations involving an output system. The radiation quantum information transmission channel is of interest because, having converted an initial qubit into the photon field, one can exploit various possibilities provided by further high-speed transformations. It is of interest to find out how rapidly information can be reconstructed after the qubit–photon field channel is used once.

The reader can find detailed computations of coherent information for this channel in [36]. Figure 4a shows the dependence of coherent information on time and the laser field action angle for a symmetrical Λ system. These results are obtained for the input qubit in the form of the state with the maximum entropy $\hat{\rho}_A = \hat{I}/2$. This qubit state corresponds to the coherent information content that is independent of individual intensities of two resonant laser fields affecting the Λ system.

One can immediately see from Fig. 4a that there exists the optimum information exchange level $R = I_c/t$ (where $t = \tau_c$) when the information channel is used periodically at duration τ_c of the exchange cycle, so that the initial state is instantly renewed after each cycle. Figure 4b demonstrates exchange rate R calculated for a symmetrical Λ system with the rates of radiative decay $\gamma_1 = \gamma_2 = \gamma$.[3] The maximum value reached by R is $R_0 = 0.178\gamma$. Thus, the atom–photon field information exchange limits the rate of using coherent information stored in Λ systems, i.e., the capacity of the Λ system–photon field coherent information channel. The order of magnitude of this rate is determined by the rate of radiative decay of an excited state, whereas its specific value depends on decay rates $\gamma_{1,2}$ of both radiative transitions of the Λ system. In the limit of a two-level radiation system, when $\gamma_1 = 0$ or $\gamma_2 = 0$, the optimum rate is equal to 0.316γ.

3. COMPATIBLE INFORMATION

For one-time mean values of quantum physical quantities, the internal quantum incompatibility manifests itself just as statistical uncertainty, which can be taken into account using the equivalent classical probability distribution. With the probabilistic measure

$$P(d\alpha) = \langle\alpha|\hat{\rho}_A|\alpha\rangle dV_\alpha \tag{9}$$

on the set of all quantum states, the mean value of any operator $\hat{A} = \sum \lambda_n |n\rangle\langle n|$ can be represented as $\langle\hat{A}\rangle = \sum \lambda_n \, dP/dV_\alpha(\alpha_n)$, where $|\alpha_n\rangle = |n\rangle$. Here, dV_α is a small volume element in the space of physically different states of a D-dimensional Hilbert space H_A ($\int dV_\alpha = D$), which is represented by the Bloch sphere in the

[3] D. Bochkarev, private communication (2001).

case of qubit, i.e, when $D = 2$ (see Section 1). Relationship (9) is the mean of the operator measure

$$\hat{E}(d\alpha) = |\alpha\rangle\langle\alpha| dV_\alpha, \qquad (10)$$

which is a special case of a nonorthogonal decomposition of unit [42], or a positive operator-valued measure (POVM) [2, 14].

Generalized quantum measurement procedures are described by POVMs. Unlike a direct measurement in the original system represented by the orthogonal decomposition of the unit, i.e., by the orthoprojective measure in H_A, a generalized quantum measurement is performed in the compound space $H_A \otimes H_a$ with the appropriate complementary state space H_a and joint density matrix $\hat{\rho}_A \otimes \hat{\rho}_a$, which contains no other information on A in addition to that contained in density matrix $\hat{\rho}_A$.

System A is characterized by uncertainty having the form of quantum incompatibility of the set comprising all of its quantum states. Generalized quantum measurement (10) transforms this uncertainty into classical statistical uncertainty of the quantitatively equivalent set of compatible events in the system $A + a$. With this representation, the coherent relationships typical of the original quantum system are transformed into the corresponding classical correlations, which have no quantum specificity. Therefore, this measurement yields a result that is not equivalent to the original system and cannot provide for further quantum transformations. This circumstance is the inevitable pay for information represented in the classical form allowing its free use. Nevertheless, initial quantum correlations are taken into account in the statistic of resultant classical states.

Let us assume that two Hilbert spaces H_A and H_B of corresponding quantum systems A and B are given and joint density matrix $\hat{\rho}_{AB}$ is specified in $H_A \otimes H_B$. In particular, A and B may correspond to the subsystems of the compound system $A + B$ specified at one and the same instant t and can be considered as the input and output of an abstract quantum channel in an actual physical system. The determining property of systems A and B is their compatibility. Hence, the joint measurement represented by two POVMs as $\hat{E}_A \otimes \hat{E}_B$ introduces no new correlations between the input and output and can be interpreted as an indicator of information input–output relationships. The corresponding joint probability distribution is

$$P(d\alpha, d\beta) = \mathrm{Tr}[\hat{E}_A(d\alpha) \otimes \hat{E}_B(d\beta)]\hat{\rho}_{AB}. \qquad (11)$$

Then, the Shannon information $I = S[P(d\alpha)] + S[P(d\beta)] - S[P(d\alpha, d\beta)]$ determines the compatible information content [32, 43].

The physical sense of compatible information depends on a specific measurement procedure and characterizes the output quantum information. This information can be obtained via two POVMs, which (in the form of classical carriers α and β) select information on the quantum state of the input, which is transmitted to the output. As in the case of one-time information, the reference system of the Schumacher scheme can serve as input A (see Figs. 2a and 3b). Then, input–output joint density matrix $\hat{\rho}_{AB}$ can be expressed in terms of the input partial density matrix and channel superoperator by formula (7).

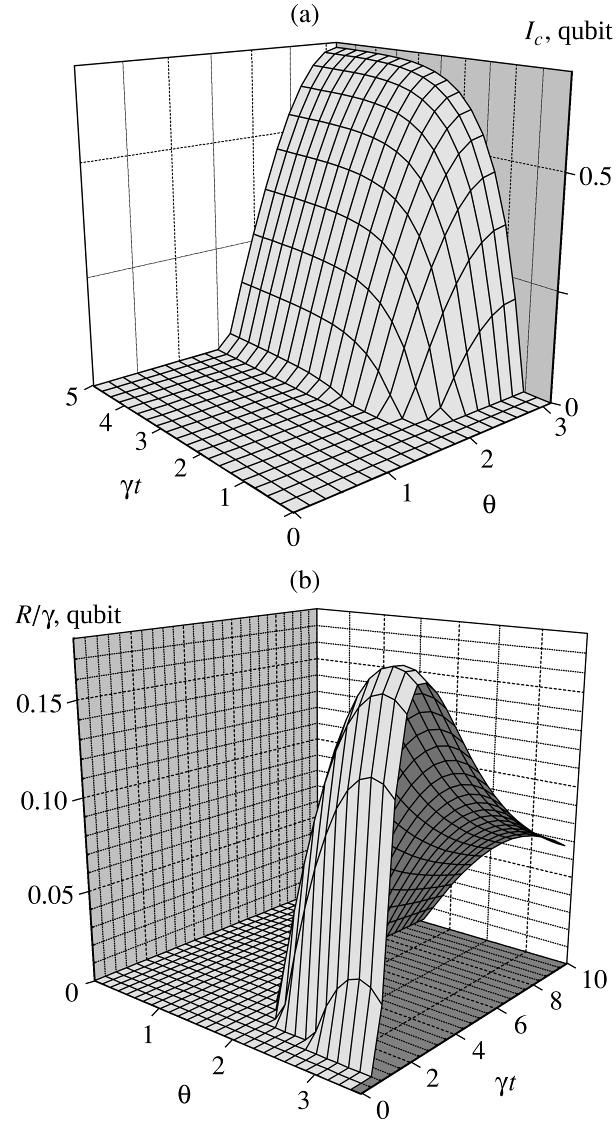


Fig. 4. (a) Coherent information I_c in a symmetrical Λ system vs. dimensionless time γt and action angle $\theta = \Omega\tau_p$ for the input state with the maximum entropy (γ is the radiation decay rate, Ω is the effective Rabi frequency, and τ_p is the exciting pulse duration) [35]. (b) Coherent information rate R vs. cycle duration t and action angle θ.

Let us consider the special case when α and β index all quantum states in H_A and H_B according to specific POVMs in forms (10). In this case, compatible information is distributed over all quantum states and asso-

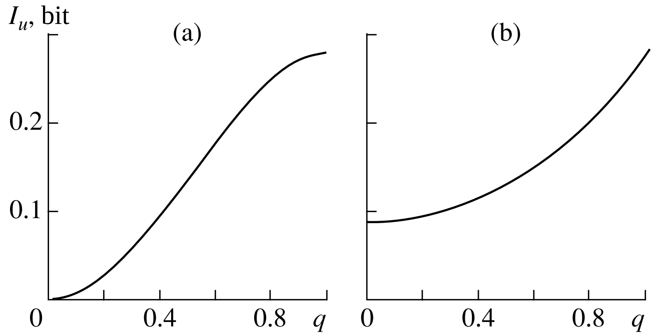


Fig. 5. Nonselected information I_u vs. entanglement parameter q for (a) a pure entangled state formed of two mutually orthogonal basis states with the weights $q/\sqrt{2}$ and $\sqrt{1-q^2/2}$ and (b) a mixed state formed of a completely entangled pure state weighted with q and a mixed state which is weighted with $1-q$ and formed of two equal-weighted pure states represented as tensor products of orthogonal basis states, so that the partial input and output entropies are equal to 1 bit for all q.

ciated with the total internal quantum uncertainty of input states, which is automatically taken into account in probability distribution (9). In particular, information that is contained in quantum correlations observed in the presence of quantum entanglement between A and B is also involved in joint probability distribution (11). In addition, here, compatible information is characterized by operational invariance [44], i.e., all noncommuting physical variables are equally taken into account in this information measure. The above representation of quantum information in terms of classical probability distributions can be interpreted as a modified quantum-mechanical representation in terms of classical physical variables applied in laser physics and discussed by R.J. Glauber in his lectures [45, 46].

A. Nonselected Information

It is natural to define information that corresponds to a generalized measurement specified in form (10) and comprises all quantum states of a system as nonselected since all quantum states are equally presented in it and quantum variables are not selected. The opposite situation occurs in the case of extremely selected information when orthogonal POVMs are used, which is typical of elementary cryptographic information exchange schemes [2]. For example, let us analyze nonselected information depending on the type of joint density matrix and its main parameter, which specifies the entanglement degree, for pure and mixed states.

(i) A pure state is specified by the wave function

$$\hat{\rho}^{(p)}_{AB}(q) = |\psi_{AB}(q)\rangle\langle\psi_{AB}(q)|,$$
$$|\psi_{AB}(q)\rangle = \sqrt{1-\frac{q^2}{2}}|1\rangle|1\rangle + \frac{q}{\sqrt{2}}|2\rangle|2\rangle \tag{12}$$

with entanglement parameter q. For the limit values $q = 0$ and 1, it yields a tensor product and completely entangled state, respectively.

(ii) A mixed state is specified by the density matrix

$$\hat{\rho}^{(m)}_{AB}(q) = (1-q)\left(\frac{1}{2}|1\rangle|1\rangle\langle 1|\langle 1| + \frac{1}{2}|2\rangle|2\rangle\langle 2|\langle 2|\right) + q|\psi_{AB}(1)\rangle\langle\psi_{AB}(1)|, \tag{13}$$

where ψ_{AB} is determined in (12). For the limit values $q = 0$ and 1, we obtain, respectively, a mixed state with purely classical correlations and a pure complete entangled state.

Computations of nonselected information are illustrated by Fig. 5. The maximum value $I_u = 0.27865$ is attained for a completely entangled state and coincides with the available information content [47] calculated in [27]. In this context, the term *availability* is considered to mean the possibility of associating with the set of all possible quantum states of information distinguishable against the quantum uncertainty background.

B. Selected Information

Selected information corresponds to generalized measurements with POVMs $\hat{E}_A$ and $\hat{E}_B$ where not all of the quantum states are equally included. The results presented below are calculated for selected information in a two-qubit system obtained using measurements $\hat{E}_A$ and $\hat{E}_B$ which combine two different types: nonselected measurements $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$ and orthoprojective measurements $\hat{E}_k$ and $U^{-1}\hat{E}_l U$ corresponding to direct measurements of orthogonal quantum states. Then,

$$\hat{E}_A(\alpha) = (1-\chi)\hat{E}_A(d\alpha), \quad \hat{E}_A(k) = \chi\hat{E}_k,$$
$$\hat{E}_B(\beta) = (1-\chi)\hat{E}_B(d\beta), \quad \hat{E}_B(l) = \chi U^{-1}\hat{E}_l U. \tag{14}$$

Here, $k, l = 1, 2$; $\hat{E}_k = |k\rangle\langle k|$; and U describes rotation of the second qubit wave function. This rotation is specified by the transformation ϑ depending on rotation angle $U(\vartheta) = \exp(i\hat{\sigma}_2\vartheta/2)$ in basis $|k\rangle$ which is proper for POVM $\hat{E}_k$ of the first qubit. Discrete output measurements (k, l) complete continual results α and β, which corresponds to new variables with an extended spectrum of values: $a = \alpha, k$ and $b = \beta, l$. In other words, we measure variables with a combined value spectrum containing a discrete component and a continuous component. These variables comprise the continuum of all wave functions and a singled-out orthogonal 2D basis. The limit cases $\chi = 0$ and 1 correspond to nonselected and complete orthoprojective measurements, respectively. The density matrix has form (13). The joint probability distribution

$$P(da, db) = \mathrm{Tr}\,\hat{\rho}_{AB}\,[\hat{E}_A(da) \otimes \hat{E}_B(db)]$$

is represented by the components

$$P(d\alpha, d\beta) = (1-\chi)^2 \langle\alpha|\langle\beta|\hat{\rho}_{AB}|\beta\rangle|\alpha\rangle dV_\alpha dV_\beta,$$
$$P(k, l) = \chi^2 \langle k|\langle l|\hat{\rho}_{AB}|l\rangle|k\rangle,$$

$$P(d\alpha, l) = (\chi(1-\chi))\langle\alpha|\langle l|\hat{\rho}_{AB}|l\rangle|\alpha\rangle dV_\alpha,$$
$$P(k, d\beta) = \chi(1-\chi)\langle k|\langle\beta|\hat{\rho}_{AB}|\beta\rangle|k\rangle dV_\beta.$$

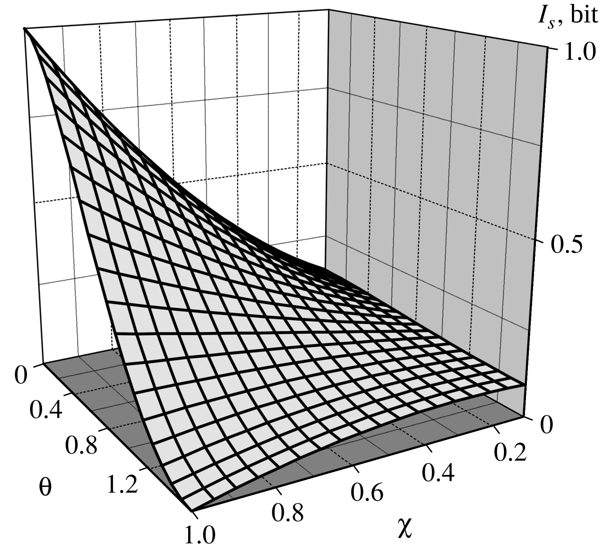
Here, the terms $P(d\alpha, l)$ and $P(k, d\beta)$ correspond to the information exchange between discrete and continual measured data for the first and second qubits. The above relationships imply the following normalization condition:

$$\iint P(d\alpha, d\beta) + \int\sum P(d\alpha, l) + \sum\int P(k, d\beta) + \sum\sum P(k, l) = 1.$$

With (13) and (14), we have two parameters: the degree of selectivity $0 \leqq \chi \leqq 1$ of a combined measurement under consideration, the relative orientation of orthoprojective measurements $0 \leqq \vartheta \leqq \pi/2$ with value extremes corresponding to the parallel and crossed orientations of the orthogonal bases of the first and second qubits, and the entanglement parameter $0 \leqq q \leqq 1$ (see Figs. 6 and 7).

The plots shown in these figures provide for the following results. The most unfavorable orientation $\vartheta = \pi/2$ reduces the selected information content down to zero at $\chi = 1$ if the selective measurement guarantees a nonzero contribution, i.e., if $\chi > 0$. At $\chi > 0$, the information content slightly depends on entanglement parameter q. The information maximum $I_s = 1$ bit is reached only for the degree of selectivity $\chi = 1$, i.e., in the case of a direct measurement.

Note a simple correspondence between nonselected and extremely selected information, which immediately follows from the physical content of the corresponding quantum measurement types and holds not only for the considered 2D example but in the general case as well. Suppose the completely selected (in our example, $\chi = 1$) measurement is performed at random. This means that we have no *a priori* information about the density matrix structure depending on the input and output information encoding, i.e., on singled-out orientation directions of the corresponding spins. Then, the information thus obtained, evidently, must be averaged over all possible orientations. The result of this averaging is exactly equal to the nonselected information content, which is due to the relationship representing the



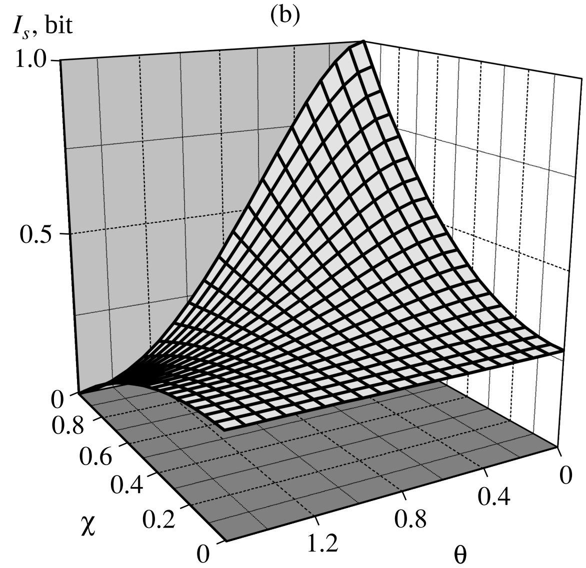


Fig. 6. Selected information I_s in a two-qubit system vs. degree of selectivity χ and relative orientation of selective measurements ϑ (a) in the absence of entanglement for quasi-classical information communication ($q = 0$) and (b) for a pure entangled state ($q = 1$).

qualitative content of input–output joint probability distribution (11) for the above two measuring procedures.

In the case under consideration, when a completely selective measurement is considered, variables α and β in (11) correspond to discrete indices of basis states k and l. With arbitrary basis wave functions $|k\rangle$ and $|l\rangle$ for input A and output B, the corresponding POVMs are $\hat{E}_k = U_A^{-1}(\alpha)|k\rangle\langle k|U_A(\alpha)$ and $\hat{E}_l = U_B^{-1}(\beta)|l\rangle\langle l|U_B(\beta)$, where quantities $U_{A,B}$ describe the rotation from the ini-

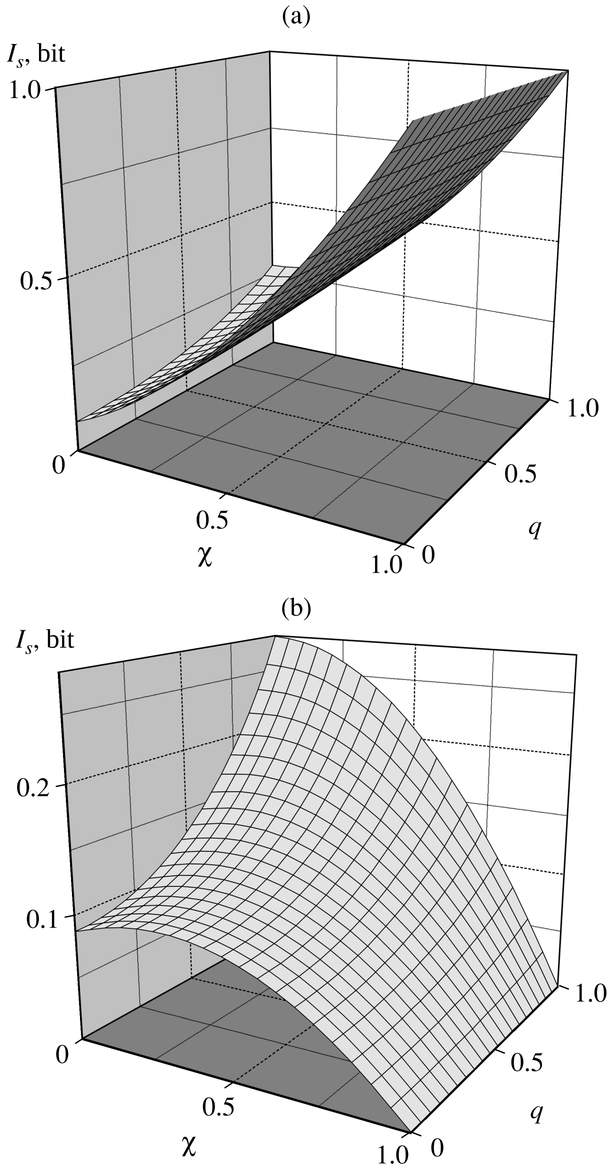


Fig. 7. Selected information I_s in a two-qubit system vs. degree of selectivity χ and entanglement parameter q for the (a) parallel ($\vartheta = 0$) and (b) crossed ($\vartheta = \pi/2$) orientations of selective measurements.

tial to measurement basis. Hence, the input–output joint probability distribution is

$$P_{kl}(\alpha, \beta) = \langle k|\langle l|U_A(\alpha)U_B(\beta)\hat{\rho}_{AB}U_B^{-1}(\beta)U_A^{-1}(\alpha)|l\rangle|k\rangle, \tag{15}$$

where α and β are the parameters of distribution P_{kl} that determine its dependence on orientations of measuring procedures. Taking into account POVM (10) and representing the expression for a nonselective measurement as a sum over projections with indices k and l of the wave functions $|\alpha\rangle = U_A^{-1}(\alpha)|0\rangle$ and $|\beta\rangle = U_B^{-1}(\beta)|0\rangle$, we again obtain dependences (15) containing α and β as information (in this case) variables. Therefore, distribution (15) over indices k and l simultaneously specifies orientation-angle probability distribution involved in the nonselective measurement procedure. Hence, the integrals describing the mean selected information content exchanged via variables k and l and the integrals describing the nonselected information exchanged via continuous variables α and β are identical. Thus, a nonselective measurement is equivalent to the set of completely selective measurements performed simultaneously for all possible orientation angles of the measurement basis. The corresponding compatible information automatically takes into account the uncertainty of the basis orientation at a completely selective measurement.

4. MEASUREMENT OF INFORMATION AVAILABLE IN A PHYSICAL EXPERIMENT

The above analysis based on generalized quantum measurements stimulates further generalizations that promise a realistic concept of information content available with a given scheme of a physical experiment, which can certainly be considered as one of the most important purposes of the quantum information theory. The main difficulty is to mathematically describe an information model of a specific experimental scheme in a sufficiently general form. To this end, first, it is necessary to mathematically define the notions of input and output, which is, actually, the most complicated task. Figure 8 demonstrates a block diagram illustrating the proposed solution.

Being affected by control interactions, the state of an object and its noisy environment changes. These interactions generate input quantum information associated either with the dynamic parameters of an object or with the set of certain quantum states that are of interest. Output information is measured at the output of the channel described by superoperator transformation $\mathcal{N}$. Superoperator measures $\mathcal{A}$ and $\mathcal{B}$ denote transformations realized by control interactions, and E_B describes the generalized quantum measurement procedure in the form complying with the POVM.

This block diagram corresponds to the typical mathematical structure of the density matrix characterizing a complex system involving two transformations ($\mathcal{A}$ and $\mathcal{B}$) which describe the control and measurement interactions, respectively:

$$\hat{\rho}_{\text{out}} = \mathcal{B}\mathcal{N}\mathcal{A}\hat{\rho}_{\text{in}}. \tag{16}$$

Here, $\hat{\rho}_{\text{in}}$ and $\hat{\rho}_{\text{out}}$ are the initial and final density matrices of the set of degrees of freedom essential within the framework of the mathematical model corresponding to a chosen experimental scheme. Superoperators $\mathcal{A}$, $\mathcal{N}$, and $\mathcal{B}$ describe physical information extraction, transmission to the input, and measurement, respectively. This Markovian structure of transformations is not the most general. For simplicity, we assume that noisy environments corresponding to each transformation are independent, so that the density matrices can be obtained from $\hat{\rho}_{in}$ and taken into account in the structure of superoperator transformations. Only due to this simplification do we have the above combination of three superoperators and the input density matrix and, as a consequence, a relatively simple mathematical representation of the information structure in terms of the corresponding decompositions of superoperators $\mathcal{A}$ and $\mathcal{B}$. However, in certain cases, it may be necessary to generalize relationship (16).

Extraction of information always involves using physical interactions described by the corresponding transformations, which are unitary only when they contain all of the degrees of freedom employed. In addition, the interaction with the noisy environment must also be included, which results in nonunitary transformations. Here, we discuss these transformations for two cases of a possible choice of desired physical information on a quantum system: (i) dynamic parameters a and (ii) quantum states $|a\rangle$.

In the first case, physical information is extracted by means of dynamic excitation of a system which is mathematically described by unitary operator $U_A(a)$. This operator may depend on control parameters c. Probabilistic measure $\mu(da)$ properly specified must take into account *a priori* information on a. Then, superoperator $\mathcal{A}$ can be represented as $\mathcal{A} = \int \mathcal{A}_a \mu(da)$ with

$$\mathcal{A}_a = \langle U_A(a) \odot U_A^{-1}(a)\rangle_E, \tag{17}$$

where the symbol $\odot$ must be replaced by the transformed density matrix and broken brackets denote averaging over the noisy environment.

In the second case, physical information can eventually be extracted in a storable form allowing for copying as a result of a certain generalized measurement corresponding to the set of positive superoperators

$$\mathcal{A}_a = \langle |a\rangle\langle a| \odot |a\rangle\langle a|\rangle_E. \tag{18}$$

In this case, the sum $\mathcal{A} = \sum \mathcal{A}_a$ is the superoperator of a generalized measurement represented using the averaged standard decomposition $\sum_i \hat{A}_i \odot \hat{A}_i^+$, which preserves the trace of a completely positive superoperator [48] with adequately specified operators $\hat{A}_i = \hat{A}_i^+ \longrightarrow |a\rangle\langle a|$. Since a may also describe continuous variables, one should use the generalized representation $\mathcal{A} = \int \mathcal{A}_a \mu(da)$ in the form of an integral with measure $\mu(da)$ guaranteeing, with allowance for idempotency

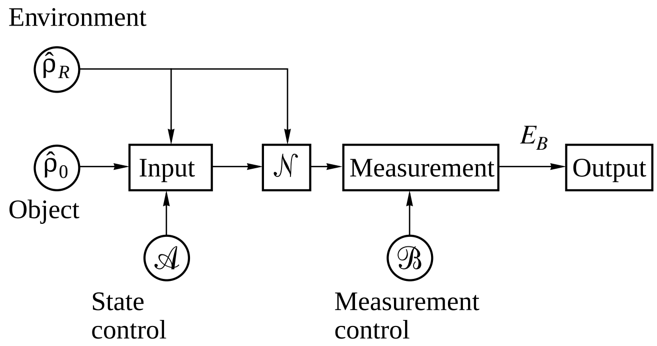


Fig. 8. Block diagram of an experimental setup.

($\hat{P}_a^2 = \hat{P}_a$) of orthoprojectors $\hat{P}_a = |a\rangle\langle a|$, that this decomposition corresponds to a certain POVM: $\int |a\rangle\langle a| \mu(da) = \hat{I}$.

In the most general form, the sets of superoperators (17) and (18) are represented using a certain positive superoperator measure (PSM): $\mathcal{A}(da) = \mathcal{A}_a\mu(da)$. This measure is a decomposition of a completely positive superoperator preserving its trace. The PSM satisfies the complete positivity ($\mathcal{A}(da)\hat{\rho} \geqq 0$) and normalization ($\mathrm{Tr}\int \mathcal{A}(da)\hat{\rho} = 1$) conditions. The latter can be represented in the equivalent form of the unit operator $\int \mathcal{A}^*(da)\hat{I} = \hat{I}$ preserved under the effect of Hermitean-conjugated PSM $\mathcal{A}^*$.

Again, it is of interest to consider the special PSM described by expression (10) with states specified in Hilbert spaces H_A and H_B which correspond to transformations $\mathcal{A}$ and $\mathcal{B}$. This PSM associates the information content experimentally extracted directly from quantum states, which yields the most explicit description of fundamental constraints due to the quantum nature of information. In this situation, the output information is represented in a stable classical form, which potentially enables numerous users to employ it simultaneously. This characteristic feature of classical information may initially be involved on default in the meaning of the term *information,* at least, when concerned with experimental physical information unlike the physical content of coherent information discussed in Section 2.

Applying the above approach to the superoperator of a measuring system $\mathcal{B} = \int \mathcal{B}(db) = \int \mathcal{B}_b \nu(db)$, where $\mathcal{B}_b$ has form (18), we can represent the input and output information as classical variables a and b describing the desired information in both cases (i) and (ii). The joint probability distribution corresponding to these variables is

$$P(da, db) = \mathrm{Tr}\mathcal{B}(db)\mathcal{N}\mathcal{A}(da)\hat{\rho}_{in}. \tag{19}$$

Evidently, this distribution is always positive and normalized to unity. It statistically associates the desired variables and output information that is extracted using an experimental setup. The information efficiency of the latter can be expressed in a quantitative form as the Shannon classical information corresponding to the above probability distribution. This quantity can be used as an optimality criterion for optimizing the experiment by available control parameters.

Note that, for choice (i), states $|a\rangle$ and $|b\rangle$ are not supposed to be mutually compatible and, in the general case, they may correspond to noncommuting variables. In the trivial limit case, they may coincide or differ by a unitary transformation, i.e., all quantum information is sent with the error probability equal to zero. In this case, the internal quantum uncertainty of this system does not enable one to use distribution (19), which establishes the one-to-one correspondence between a and b. If the states belong to physically different subsystems, they, nevertheless, may contain quantum correlations due to the corresponding superoperator transformation of channel $\mathcal{N}$. The simplest example is the superoperator $\mathcal{N} = U_{AB} \odot U_{AB}^{-1}$, which describes a unitary transformation entangling the input and output states.

Control parameters c may be fixed or chosen from a certain set $c \in \mathbf{C}$ of necessary values. In the latter case, information can be optimized according to the above criterion. The presence of unknown *a priori* distribution $\mu(da)$ for dynamic parameters a in this information structure is not caused by quantum specific features of the problem, i.e., the problem of *a priori* uncertainty has to be treated by the methods employed in the classical theory of optimal statistical decisions [49]. In the most general case, transformations $\mathcal{B}_b$ (see (18)) of a measuring system can be described with an arbitrary PSM.

Consequently, PSMs $\mathcal{A}(da)$ and $\mathcal{B}(db)$ cover a very wide range of possible types of object quantum state control when the above quantum measurement procedure is implemented in the experimental procedure under study.

CONCLUSION

In this paper, the most general classification of quantum information is proposed. It is based on compatibility/incompatibility of the input and output states of a quantum channel. According to this classification, all possible types of information are categorized as classical, semiclassical, coherent, and compatible types of information.

The physical content of coherent information is the amount of information contained in internally incompatible states that is exchanged between two systems and quantified as the entanglement preserved between the output and the reference system. The entanglement with the latter is used to measure information at the input and output of a quantum channel specified by a superoperator transformation. Here, we introduce the concept of one-time coherent information with the information channel represented by the corresponding joint density matrix. With this concept, two approaches to determining quantum information are adequately associated with each other. According to the first approach, the channel is specified by a superoperator transformation of the input density matrix, and according to the second, by the joint input–output density matrix. The coherent information exchange rate calculated for the channel between a Λ system and the field of free photons yields the upper bound equal to 0.178γ for a symmetric Λ system and 0.316γ in the absence of this constraint.

The necessity of introducing compatible information as an adequate characteristics of quantum information exchange between two compatible systems of quantum states is justified. Compatible information is expressed in terms of the classical information theory despite the presence of internal quantum incompatibility of states in contrast to coherent information, which principally cannot be reduced to classical representations. Nevertheless, the determination of coherent information is genetically related to compatible information because it is based on distinguishing a pair of compatible systems similar to the input and output systems involved in the analysis of compatible information. One of them is the reference system, and the second is the input or output. Thus, the presence of mutually compatible sets of quantum states is necessary for quantifying information of each of the above types. However, when interacting systems are considered, different types of information exhibit qualitatively different types of behavior. The reason for this is that, unlike coherent information, the presence of compatible information may be due to both purely quantum and classical input–output correlations. Particularly, in the Dicke problem, this circumstance results in possible existence of compatible information (in contrast to coherent information) after the short-lived collective Dicke state decays.

Selection of quantum states is shown to be principal for obtaining a useful compatible information content. It is found that nonselected information is equivalent to completely selected information which averaged over all possible orientations the orthogonal bases of input and output complete quantum measurements, realizing complete selection of quantum states.

It is shown that mutual and internal compatibility, i.e., the property of input and output quantum information being classical, is a natural limitation of the physical content of the information flow in an experimental setup. This circumstance enables one to introduce a sufficiently general unified mathematical structure corresponding to a chosen scheme of an actual physical experiment and to quantify its information efficiency.

In this situation, the information exchange between subsystems preparing quantum information and a measuring device is described by the probabilistic correspondence between classical variables determining the physical parameters of a quantum system under study and measured output variables. Quantum information generation and readout are represented in the general mathematical form with two PSMs. This mathematical representation of quantum information exchange realized experimentally looks promising for applying the quantum information theory to physical experiments. The approaches proposed in this paper additionally justify the general statement on the physical concept of quantum information mentioned in the main paper heading [50].

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