Laser Physics, Vol. 11, No. 12, 2001, pp. 1324–1331.

Compatible Information as a Natural Information Measure of a Quantum Channel

B. A. Grishanin and V. N. Zadkov

Physical Faculty, International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, Moscow, 119899 Russia

e-mail: ghrishan@comsim1.phys.msu.su

Received June 7, 2001

Abstract—The Shannon information amount corresponding to two independent generalized measurements of all possible quantum states at the input and output is proposed as a natural quantitative characteristic of mutual information stored in two compatible sets of quantum states considered as an input and an output. We analyze the physical content of this information measure and its relation to other measures, such as the Holevo information and coherent information. Using an example of two two-level systems, we reveal and discuss the main essential properties of compatible information in the absence and in the presence of selection of states being measured.

1. INTRODUCTION

There are two main approaches to the quantitative description of information in quantum channels. One of these approaches is based on the Holevo information [1, 2] and is related to a semiclassical version of quantum generalization [3] of the classical theory of optimal decisions [4] and mathematical representation of a quantum measurement in the form of a positive operator-valued measure (POVM) [5]. Classical Shannon information [6] applied to classical input variables of a semiclassical channel—indices α of possible input quantum states—and quantum output variables is an adequate information measure in this case. Another approach is based on coherent information [7, 8], introduced as a characteristic of a fully quantum channel. This parameter is described as the difference between the total output entropy and the exchange entropy added to the output from a reservoir interaction with which in information transfer through a channel describes the mechanism behind the appearance of noise and the corresponding information losses.

There is a substantial, qualitative difference between these two types of quantum information, since coherent information vanishes if the transfer of this information is elated to measurement [9], while the Holevo information is directly related to a measurement procedure. Coherent information is, in fact, an array of purely quantum information transferred from the input to the output, i.e., information destroyed at the input, since copying of quantum information is impossible [10, 11]. A semiclassical information at the input is associated with classical parameters, which allow copying. In this paper, we point out that there is another important and natural type of information in between these two limiting cases of information—compatible information. Information of this type is based on two simultaneous, i.e., compatible generalized measurements, performed at the input and at the output of a quantum channel. The term compatibility is employed here in the traditional quantum-mechanical sense as the possibility of applying classical logic to the relevant pair of quantum events a and b, chosen from all the events related to some quantum systems A and B, or a pair of quantum physical quantities described by operators $\hat{a}$ and $\hat{b}$. Mathematically, this is expressed as the absence of nonorthogonality in the set of subspaces corresponding to a and b or the commutative nature of operators $\hat{a}$ and $\hat{b}$, guaranteed in accordance with basic principles of quantum mechanics applied to the same moment of time. Compatible generalized measurements are represented in the form of the relevant tensor product $\hat{E}_A \otimes \hat{E}_B$ composed of $\hat{E}_A$ and $\hat{E}_B$ POVMs describing measurements at the input and output, respectively. In particular, when these measurements are related to classical sets $\mathcal{A} \ni \alpha$ and $\mathcal{B} \ni \beta$ of indices of all quantum states $\psi_\alpha \in H_A$ and $\psi_\beta \in H_B$ of the input and output quantum Hilbert spaces, the result of this measurement depends only on the input–output density matrix and characterizes the information correspondence between all the quantum states at the input and output. Starting with the relevant sets of wave functions ψ_α and ψ_β with known joint input–output matrix $\hat{\rho}_{AB}$, we arrive at the following expression for the corresponding joint probability distribution of the results of an input–output measurement:

$$P(d\alpha, d\beta) = \langle\alpha|\langle\beta|\hat{\rho}_{AB}|\beta\rangle|\alpha\rangle dV_\alpha dV_\beta, \tag{1}$$

where $dV_{\alpha,\beta}$ describe volume differentials in the spaces of variables α and β. This probability distribution corresponds in a natural way to a Shannon information amount, which defines an array of compatible input–output information.

Mathematically, the proposed information measure does not involve anything new. Moreover, this information measure inevitably appears in a sufficiently close form in any discussion of classical information related to a quantum channel (e.g., see [12, 13]). However, a qualitatively new aspect is the analysis of the possibility to apply this measure as the most general characteristic of a quantum channel with compatible spaces of input and output states with the use of the above-specified structure of sets of the considered manifolds of input and output states. This structure is natural for the discussion of quantum systems in terms of the distinguishability of quantum states [13]. Compatible information is the most general characteristic of a quantum system where the input and the output are represented as mutually compatible sets of quantum events governed by a direct product of Hilbert spaces, $H_{AB} = H_A \otimes H_B$. The equivalence of contributions from all the states of these spaces makes this characteristic operation-invariant. In other words, this quantity ensures an equivalent inclusion of the information related to various noncommuting input and output quantum variables [14].

We should emphasize that, when we consider the time evolution of a quantum system, the above-discussed compatibility implies that either single-moment sets of quantum events or, at least, events that occur at different moments of time t_1 and t_2, but that do not become entangled within the time (t_1, t_2), even if quantum entanglement took place before the moment t_1, are employed as an output and an input. Otherwise, an incompatibility (nonorthogonality) related to a partial transfer of quantum uncertainty from the input to the output of the channel appears between the input and output states. Thus, to deal with a compatible information only in the case of an input and output shifted in time, we should associate the input information with some reference states that remain unchanged in the course of time.

In this paper, we discuss the physical meaning of the input and output spaces of quantum states and the joint density matrix for a specific physical example of a fully quantum channel. We will calculate the dependences of compatible information for such a channel on the most important parameters of a quantum measurement.

2. DEFINITIONS AND GENERAL RELATIONS

Suppose that two Hilbert spaces of states H_A and H_B of quantum systems A and B are initially defined. These spaces may correspond, in particular, to subsets of a two-component system $A + B$ or an input and output defined in some other way for an abstract quantum channel implemented in a specific physical system (possible classification of various types of quantum channels can be found in [9]). To determine the quantum information measure, we should define the averaging procedure for any pair of Hermitian operators $\hat{A}$ and $\hat{B}$ in H_A and H_B, respectively. For this purpose, we simply assume that the joint input–output density matrix $\hat{\rho}_{AB}$ is known, so that $\langle \hat{A} \otimes \hat{B} \rangle = \mathrm{Tr}(\hat{A} \otimes \hat{B})\hat{\rho}_{AB}$. We emphasize that the density matrix $\hat{\rho}_{AB}$ is assumed to be self-conjugate and positive even in the case of the most general type of the considered channel, as opposed to generalized multimoment density matrix, appearing in the calculation of multimoment means of a quantum random process [15–17]. Introducing two relevant POVMs $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$ to represent two generalized measurements in A and B, we arrive at a joint classical input–output probability distribution:

$$P(d\alpha, d\beta) = \mathrm{Tr}(\hat{E}(d\alpha) \otimes \hat{E}(d\beta))\hat{\rho}_{AB}. \qquad (2)$$

It is important to emphasize here that, although these two measurements are compatible due to the fact that the output variables being measured commute with the input variables, these measurements store information in the relevant quantum correlations only if the joint density matrix involves quantum entanglement. This circumstance is of fundamental importance for the schemes using specific properties of quantum information in two-body systems [5].

The probability distribution (2) is directly related to the Holevo information for a semiclassical channel defined by an ensemble of mixed states $\hat{\rho}_\mu$ with probabilities p_μ for a measurement at the output represented by a POVM $\hat{E}(d\nu)$. Starting with this set of characteristics, we can pass to characteristics employed in Eq. (2) by introducing a quantum input instead of the classical input α with the use of the input–output density matrix $\hat{\rho}_{AB} \sim \sum \hat{E}_0(d\mu) \otimes \hat{\rho}_\mu$, where $\hat{E}_0(d\mu)$ describes the orthoprojector measure defining a direct quantum measurement in an appropriate quantum system A. It can be easily seen then that the semiclassical input–output joint distribution $P(d\mu, d\nu) = p_\mu \mathrm{Tr}\hat{\rho}_\mu \hat{E}(d\nu)$ coincides with its fully quantum representation (2) if $\hat{E}(d\mu) \sim \hat{E}_0(d\mu)$. To make this correspondence less formal, it is sufficient to recall that, with an adequate description, all the classical systems are also subject to the laws of quantum physics. Therefore, there always exists also a fully quantum physical realization of the introduced operators as long as the considered classical description is physically adequate.

Although the distribution (2) is substantially classical, it satisfies the requirement of the possibility of mapping the quantum-mechanical averaging of functions of the type $f(\hat{A} \otimes \hat{B})$ of systems A and B (which can be generally supplemented with reservoir variables):

$$\langle f(\hat{A} \otimes \hat{B}) \rangle = \iint f(\lambda_A, \lambda_B) \langle \hat{E}(d\lambda_A) \otimes \hat{E}(d\lambda_B) \rangle.$$

This is exactly what is of interest in any physical experiment, when the detection system gets an information from one (output) subsystem B, and the purpose of experiments is to make quantitative conclusions concerning variables of the other (input) subsystem A. The main idea of this work is that this distribution provides an opportunity to describe the information capabilities of a quantum channel, since an experimenter in the considered experimental situation, in fact, always deals with this distribution even if no real measuring procedure is performed and only potentially possible quantum-mechanical means are discussed. To develop the corresponding information description, it is sufficient to consider a unified most detailed POVM

$$\hat{E}(d\nu) = |\nu\rangle\langle\nu| dV_\nu, \tag{3}$$

where $|\nu\rangle$ represents all possible wave functions in the Hilbert space and dV_ν describes the volume differential in the space of variables ν, instead of all possible POVMs corresponding to $\hat{E}(d\lambda)$ in the Hilbert space H. Positive operator-values measures corresponding to specific physical variables $\hat{F} = \int \lambda \hat{E}_0(d\lambda)$ can be then constructed in the form

$$\hat{E}(d\lambda) = \int_{\lambda(\nu) \in d\lambda} |\nu\rangle\langle\nu| dV_\nu.$$

Thus, using the unified POVM (3), one still has an opportunity of characterizing means for any pair of physical quantities.

For a given quantum channel represented here by the joint input–output density matrix $\hat{\rho}_{AB}$, we introduce a quantum analogue of the Shannon information—an unselected information I_u. This information is defined as the Shannon information corresponding to the joint input–output probability distribution (2) for measurements of the form (3):

$$I_u[\hat{\rho}_{AB}] = S[P(d\alpha)] + S[P(d\beta)] - S[P(d\alpha, d\beta)], \tag{4}$$

where $P(d\alpha) = \int_\beta P(d\alpha, d\beta)$, $P(d\beta) = \int_\alpha P(d\alpha, d\beta)$ and α and β describe the indices of all the wave functions in the Hilbert spaces H_A and H_B, respectively, corresponding to two unified POVMs $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$ in H_A and H_B. Since α and β exhaust all the states $|\alpha\rangle$ and $|\beta\rangle$ of the Hilbert spaces H_A and H_B, they also contain the information regarding the quantum mature of the channel. If some arbitrarily defined POVMs $\hat{E}_A$ and $\hat{E}_B$ are considered instead of unified POVMs $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$, we generally obtain a selected information

$$\begin{aligned} &I_s[\hat{\rho}_{AB}, \hat{E}_A, \hat{E}_B] \\ &= S[P_A(\hat{E}_A)] + S[P_B(\hat{E}_B)] - S[P_{AB}(\hat{E}_A, \hat{E}_B)], \end{aligned} \tag{5}$$

which depends on the choice of the special measuring procedure, corresponding, in particular, to some selected physical variables. Here, P_A, P_B, and P_{AB} describe the partial and joint probability distributions corresponding to the chosen POVMs.

In terms of the definitions given above, the mathematical representation of a channel somewhat differs from the representation typical of the cases when the above-mentioned types of information are considered. In those cases, a channel is introduced directly as a transformation implemented by the channel, i.e., as a conditional output–input distribution $P(d\beta|\alpha)$ in the classical Shannon theory, a conditional output–input ensemble of quantum states $\hat{\rho}_\alpha$ in the semiclassical theory, or the relevant superoperator transform of the input density matrix $\mathcal{N}$ in the theory of coherent information. However, the information considered in this paper does not require a direct definition of this channel characteristic, but is based on the joint density matrix $\hat{\rho}_{AB}$ and the corresponding distribution $P(d\alpha, d\beta)$, which includes, along with the input distribution $P(d\alpha)$, the conditional distribution $P(d\beta|d\alpha) = P(d\alpha, d\beta)/P(d\alpha)$. Thus, the quantum application of the classical Shannon discussed in this paper is even more symmetric than its classical prototype.

The unselected information (4) is independent of local unitary transformations of the two-body density matrix $\hat{\rho}_{AB}$. Being applied to $\hat{\rho}_{AB}$ in calculations of joint distributions $P(d\alpha, d\beta)$, these transformations can be equivalently applied to POVMs $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$. Due to the completeness of the states ψ_α and ψ_β, this is manifested in the form of the geometric transform in spaces of values of α and β, leaving the information amount unchanged.

The situation is different in the case of selected information, since unitary transformations of POVMs $\hat{E}_A$ and $\hat{E}_B$ may substantially modify the distribution $P_{AB}(\hat{E}_A, \hat{E}_B)$. An important aspect here is the possibility to apply various transformations on $\hat{E}_A$ and $\hat{E}_B$ to distinguish between orthogonal bases of quantum states for systems A and B having different correlation properties if the state $\hat{\rho}_{AB}$ corresponds to information exchange through certain physical variables. Conversely, if unitary transformations are reduced to a simple commutation within basis states, the information amount remains unchanged. Another important aspect is that, in contrast to classical information, the use of POVMs with a reduced set of considered argument values does not necessarily decreases the amount of information if nonorthogonal states are involved. This is due to the fact that the addition of nonorthogonal states may lower the amount of accessible information, rather than to increase this amount, since the free entropy (information) accessible for exchange [13] may remain nonincreasing with such an addition, while the total uncertainty grows. Moreover, the optimal choice of the reduced POVM in the case of entangled states allows the information amount to be increased due to the addition of quantum fluctuations correlated at the input and output to purely classical relations (see Section 4.2).

3. AN EXAMPLE OF A PHYSICAL QUANTUM CHANNEL

Let us show how the definition of compatible information can be applied to a physical model of a channel with quantum input and output. First, we should reconstruct the joint density matrix $\hat{\rho}_{AB}$ in the case when the channel is defined by the relevant transform superoperator $\mathcal{N}$ [7]. In fact, this procedure is reduced to the determination of the true physical input of the channel corresponding to its abstract definition in the form of the density matrix. Similar to any density matrix in quantum theory, the input density matrix $\hat{\rho}^Q$ in [7], as it is well known, cannot be a primary object of quantum theory. This matrix can be constructed from more fundamental concepts: either from a pure state of some closed quantum system or as an equivalent representation of a physically motivated mixed ensemble of pure states. Correspondingly, the procedure of the so-called purification of a quantum state, which represents the input density matrix as an equivalent of a pure state in an expanded system $Q + R$, in fact, implies that we come back to the rigorously quantum prototype of the density matrix, mapped in the additional reference system R by quantum fluctuations. Since the transform implemented by the channel on the system Q changes its state, $Q \longrightarrow Q'$, where $Q' = B$ in our notations, the true input in terms of the above-described concepts is understood as an invariant state of the reference system R that is not covered by the transform of the channel $\mathcal{N}$.

To restrict ourselves to the framework of input description employed in the theory of coherent information, we should use, in accordance with [9], a special choice of the reference system in the form $R = A$. In this case, we find that

$$\hat{\rho}_{AB} = \sum \sqrt{p_i p_j}(\mathcal{N}|i\rangle\langle j|) \otimes |i^*\rangle\langle j^*|, \quad (6)$$

where $\mathcal{N}$ is the channel superoperator, $p_{i,j}$ and $|i, j\rangle$ are the eigenvalues and the eigenvectors of the input density matrix $\hat{\rho}^Q$, and asterisks stand for the eigenvectors of the complex-conjugate (mirror-reflected) input density matrix $\hat{\rho}^{Q*}$. The latter matrix describes the initial state of the physical input A, which transfers its quantum information to the intermediate input $A = A^*$, represented by the input density matrix $\hat{\rho}^Q$ through a pure two-component state $\psi_{AA^*} = \sum \sqrt{p_i}|i\rangle|i^*\rangle$. The von Neumann entropy $S[\hat{\rho}^Q] = S[\hat{\rho}^{Q*}]$ describes the initial entanglement of the system $A + A^*$, and the corresponding coherent information $I_c = S[\hat{\rho}_B] - S[\hat{\rho}_{AB}]$ can be interpreted as saved entanglement, which coincides with the initial entanglement, $S[\hat{\rho}_A] = S[\hat{\rho}_B]$, in the case of a noise-free channel (i.e., a channel implementing an identity transform, $\mathcal{N} = \mathcal{I}$), generally decreasing after the transformation implemented by a noisy channel.

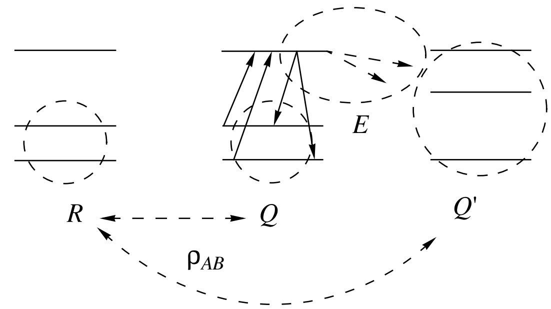

Fig. 1. The structure of a quantum information system in the case of two Λ systems with two radiatively stable levels employed as a qubit (with the use of notations of [20] for constituent subsystems). In the analysis of compatible information, two entangled qubits R and Q are considered as an input and an output of a quantum channel. The qubit Q is also considered as an input of the qubit–photon field Q' channel [18]. The photon field stores coherent information concerning the input qubit. Entanglement with the qubit R, which is considered as a reference system, is the content of this coherent information. To read out this coherent information, two resonant laser pulses excite a qubit Q, leading to a radiative decay with emission of two photons and excitation of the three-level subsystem Q' of the photon field, which serves as an output B for the channel. Radiative decay occurs in the system $Q + E$, where E is a reservoir formed by the upper excited state of the Λ system and the states of the photon field that are not included in the subsystem Q'.

A characteristic example illustrating the specific features of quantum-information transfer in physical systems is the model of information exchange between a qubit of a Λ system and a vacuum photon field [18] (see Fig. 1). The ground radiatively stable two-level states of Λ systems are employed as carriers of information qubits, which are believed to be promising carriers of quantum information potentially suitable for the storage of quantum information and implementation of quantum logic gates (see, e.g., [19] and references therein). The channel is formed by a qubit Q, which is described in theory [20] as an input channel and which is represented by the input density matrix $\hat{\rho}^Q$, and the output Q'. The latter is represented here by a three-dimensional Hilbert space formed by vacuum and two one-photon states of the radiation field excited with two dipole-active transitions from the excited state to the ground state of the Λ system, which is represented by a two-level system with levels split in their energy.

The process of information transfer involves two laser fields transforming the initial qubit into the excited state, which is manifested in the form of two emitted photons, serving as carriers of quantum information. The excited atomic state and the other degrees of freedom of the photon field work as a reservoir. The meaning of the process of laser excitation is twofold. On the one hand, this process provides an encoding transform, ensuring a nonzero level of information exchange between the information qubit and the photon field. On the other hand, it acts beyond the space of states of the input qubit, because the qubit is associated only with the ground state. If we still associate the initial quantum information with the complete three-dimensional space of the Λ system, then this process becomes reversible and is represented by the relevant unitary input–input transform. Otherwise, it would be reasonable to include this process in the channel superoperator $\mathcal{N}$ as an additional input–Λ-system transform at the input of the Λ-system–photon-field channel. Consequently, we have some degrees of freedom in choosing the terminology employed to describe the information significance of this transform, associated with the choice of the input and output. The meaning of the reference system R [20] is also twofold. From the mathematical viewpoint, this system is introduced only to express the mixed initial state $\hat{\rho}^Q$ through a pure entangled state ψ_{QR} of a broader system $Q + R$, represented above as ψ_{AA*}, in order to relate the amount of quantum information stored in the form of entanglement, $S[\hat{\rho}^Q] = S[\hat{\rho}^R]$, to this state. However, as it follows from the example considered above, this reference system bears also some real physical meaning as a subsystem of a real two-qubit system, serving as a basis for the organization of quantum computations [21]. The entanglement between the photon field and the reference two-level system R is then the specific physical content of coherent information I_c.

4. ANALYSIS OF COMPATIBLE INFORMATION IN A SYSTEM OF TWO QUBITS

To calculate the input–output density matrix and the corresponding amount of compatible information for specific physical systems, one should employ the technique of calculations described in [9, 18]. A channel defined in the form (6), which is symmetric with respect to the input and output, involves both a purely quantum information exchange by means of quantum entanglement and exchange through classical correlations between the input and output. Some internal correlations between the input and output that would be inherent in the measurement procedure described by the POVM $\hat{E}(d\alpha) \otimes \hat{E}(d\beta)$ are absent by definition. Correlations may be only due to a quantum entanglement and/or classical input–output correlations, described by the two-body density matrix $\hat{\rho}_{AB}$.

Below, we will consider a physical example of an information system illustrated in Fig. 1. Coherent information for a channel involving a radiatively stable qubit and the photon field was calculated in [18]. It is appropriate to start the analysis of compatible information by considering a simpler information subsystem, where quantum information is not transformed into the photon field, but is stored in the form of radiatively stable qubits of two Λ systems. The output Q' of a given channel then coincides with the input Q, and the density matrix $\hat{\rho}_{AB}$ describes the joint state of two qubits generally corresponding to an entangled state. Physical methods for the creation of such states are currently in the stage of intense development [22]. As for the considered generalized measurements, their physical realization is only guaranteed by the existence of their representation in the form of a standard quantum-mechanical measurement in the relevant extended space of states, and it is too early to discuss some specific methods of physical implementation of such measurements. Because of this reason, the practical meaning of the calculated compatible information is not reduced to the possibility of using this information to transmit large arrays of arbitrary messages, as it is usually the case in classical theory [6]. However, being related to the classical input–output information stored in a quantum system, this quantity, similar to the classical theory, retains its significance as the most universal quantitative characteristic of the information on the results of measurements at the input obtained from the results of measurements at the output, defining the correlation degree of these results. Moreover, even in the absence of measurements, this information characterizes the correlation degree for the values of any physical variables $\hat{A}$ and $\hat{B}$ at the input and output defined by operators in the spaces of states $H_A \otimes H_1$ and $H_B \otimes H_2$ of the input and output supplemented with other independent degrees of freedom.

Of fundamental importance is the consideration of variables of a special type—nonselective variables. Such variables are described by operators in a composite space $H_A \otimes H_1$ ($H_B \otimes H_2$) having a nondegenerate spectrum and sets of the relevant orthogonal eigenfunctions $\|\alpha\rangle\rangle$, which mutually uniquely correspond to all the wave functions $|\alpha\rangle$ of the initial space. These eigenfunctions describe the corresponding entangled states in $H_A \otimes H_1$. Note that, if necessary, the supplementary space H_1 should be infinitely dimensional, while the above-specified eigenfunctions should be nonnormalizable. If the state $|A\rangle$ in the supplementary subset H_1 is chosen in such a way that the projection of functions $\|\alpha\rangle\rangle$ on this state yields the corresponding wave function $|\alpha\rangle = \langle A\|\alpha\rangle\rangle$, then the above-specified orthogonal basis of entangled states uniquely corresponds to all the

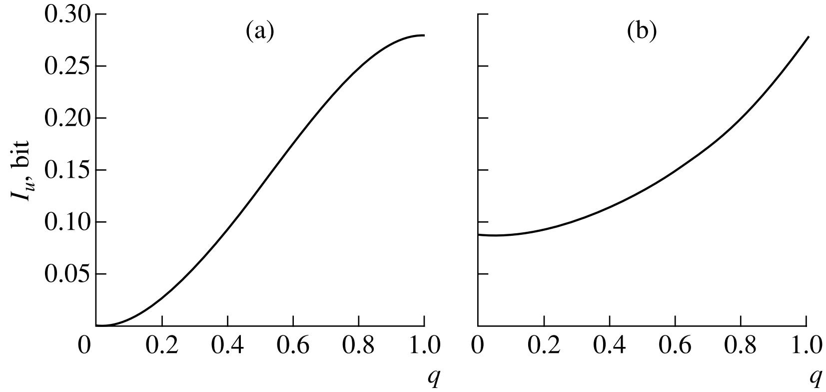

Fig. 2. Dependence of unselected information (of the Alice–Bob type [5]) on the entanglement parameter for (a) a pure entangled state formed by two mutually orthogonal basis states with the weights $q/\sqrt{2}$ and $\sqrt{1-q^2/2}$ and (b) a mixed state formed by a totally entangled pure state taken with the weight q and a mixed state taken with the weight equal to $1 - q$ and composed of two equally weighted pure states in the form of tensor products of orthogonal basis states in such a way that the partial input and output entropies are equal to 1 bit for all q.

nonorthogonal wave functions $|\alpha\rangle$. Thus, this basis equivalently represents the quantum uncertainty inherent in a nonorthogonal set of all the wave functions $|\alpha\rangle$ in the form of a completely equivalent statistical uncertainty that arises when only the variables of the initial set H_A are considered and that belongs to the orthogonal basis, i.e., a set of classically compatible events – entangled states of the composite system. Thus, a complete ensemble of incompatible quantum events is replaced by a set of compatible entangled states in an extended system. The mean values of nonselective input and output variables are expressed through the joint input–output probability distribution $P(d\alpha, d\beta)$, where all possible input and output states, in fact, serve as arguments. It is this consideration that is most adequate to the needs of quantum computations and cryptography.

4.1. Unselected Information

As an example, we will analyze the dependence of the unselected information on the type of the joint density matrix and its main parameter, determining the entanglement degree, for the cases of pure and mixed states.

(a) A pure state is defined by the wave function

$$\hat{\rho}^{(p)}_{AB}(q) = |\psi_{AB}(q)\rangle\langle\psi_{AB}(q)|,$$
$$|\psi_{AB}(q)\rangle = \sqrt{1-\frac{q^2}{2}}|1\rangle|1\rangle + \frac{q}{\sqrt{2}}|2\rangle|2\rangle, \tag{7}$$

with an entanglement parameter q. For the limiting values $q = 0$ and 1, this wave function gives, respectively, a tensor product and a totally entangled state.

(b) A mixed state is defined by the density matrix

$$\hat{\rho}^{(m)}_{AB}(q) = (1-q)\left(\frac{1}{2}|1\rangle|1\rangle\langle1|\langle1| + \frac{1}{2}|2\rangle|2\rangle\langle2|\langle2|\right) + q|\psi_{AB}(1)\rangle\langle\psi_{AB}(1)|. \tag{8}$$

For two limiting values $q = 0$ and 1, we arrive, respectively, at a mixed state with purely classical correlations and a pure totally entangled state.

The results of calculations for the unselected information are presented in Fig. 2. The maximum value $I_u = 0.27865$ bit is achieved for a totally entangled state and coincides with the amount of accessible information [23], calculated in [13]. The accessibility of information is understood in this context as the possibility of associating the information distinguishable against the background of quantum uncertainty with the set of all possible quantum states.

4.2. Selected Information

Selected information in a system of two qubits was calculated with the use of measurements $\hat{E}_A$ and $\hat{E}_B$ resulting from a combination of two types of measurements: unselected measurements $\hat{E}(d\alpha)$ and $\hat{E}(d\beta)$ and orthoprojector measurements $\hat{E}_k$ and $U^{-1}\hat{E}_l U$, which correspond to a complete measurement of quantum states. Correspondingly, we have

$$\hat{E}_A(\alpha) = (1-\chi)\hat{E}_A(d\alpha), \quad \hat{E}_A(k) = \chi\hat{E}_k,$$
$$\hat{E}_B(\beta) = (1-\chi)\hat{E}_B(d\beta), \quad \hat{E}_B(l) = \chi U^{-1}\hat{E}_l U. \tag{9}$$

Here, $k, l = 1, 2$; $\hat{E}_k = |k\rangle\langle k|$; and U describes the rotation of the wave function of the second qubit specified in the form of a transform $U(\vartheta) = \exp(i\hat{\sigma}_2\vartheta/2)$,

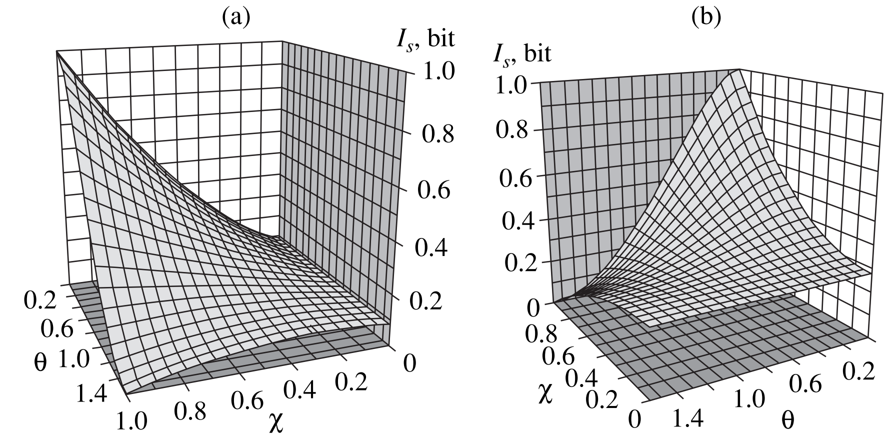

Fig. 3. Dependence of selected information in a system of two qubits on the selectivity degree χ and the relative orientation of selective measurements ϑ (a) in the absence of entanglement in the case of a quasi-classical information relation, $q = 0$, and (b) for a pure entangled state, $q = 1$.

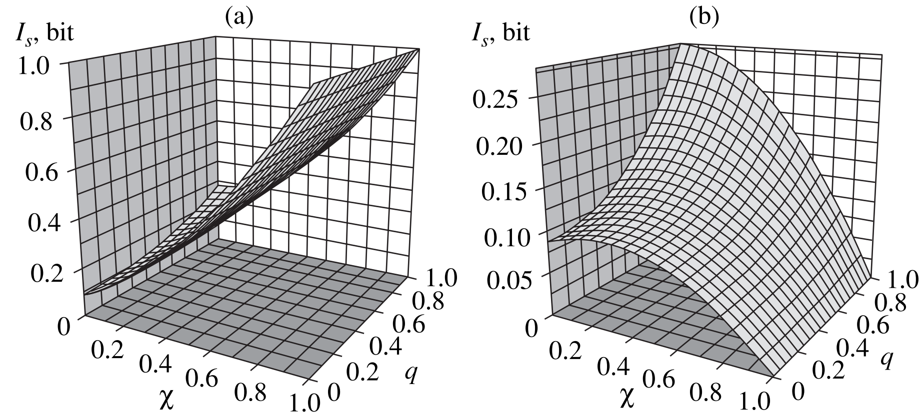

Fig. 4. Dependence of selected information in a system of two qubits on the selectivity degree χ and the entanglement parameter q: (a) for a parallel orientation of selective measurements, $\vartheta = 0$, and (b) for a crossed orientation of selective measurements, $\vartheta = \pi/2$.

depending on the rotation angle ϑ and defined by this expression in the basis $|k\rangle$, which is an eigenbasis for POVMs $\hat{E}_k$ of the first qubit. Discrete results of measurements at the output, k, l, supplement continual results α and β, which corresponds to new variables with an extended spectrum of values: $a = \alpha, k$ and $b = \beta, l$. In other words, we measure variables with a combined spectrum of values, involving discrete and continuous components, including the continuum of all the wave functions and the selected orthogonal two-dimensional basis. The density matrix is written in the form of Eq. (8). The joint probability distribution is written as $P(da, db) = \mathrm{Tr}\hat{\rho}_{AB}[\hat{E}_A(a) \otimes \hat{E}_B(b)]$ and is represented by the components

$$P(d\alpha, d\beta) = (1-\chi)^2\langle\alpha|\langle\beta|\hat{\rho}_{AB}|\beta\rangle|\alpha\rangle d\alpha d\beta,$$

$$P(k, l) = \chi^2\langle k|\langle l|\hat{\rho}_{AB}|l\rangle|k\rangle,$$

$$P(d\alpha, l) = \chi(1-\chi)\langle\alpha|\langle l|\hat{\rho}_{AB}|l\rangle|\alpha\rangle d\alpha,$$

$$P(k, d\beta) = \chi(1-\chi)\langle k|\langle\beta|\hat{\rho}_{AB}|\beta\rangle|k\rangle d\beta.$$

Here, the terms $P(d\alpha, l)$ and $P(k, d\beta)$ correspond to information exchange between the discrete and continual results of measurements of the first and second qubits. Keeping in mind Eqs. (8) and (9), we have three parameters here: the selectivity degree, $0 \leqq \chi \leqq 1$, of the combined measurement under consideration; the relative orientation of orthoprojector measurements, $0 \leqq \vartheta \leqq \pi/2$, with limiting values corresponding to the parallel and crossed orientations of orthogonal bases of the first and second qubits; and the entanglement parameter, $0 \leqq q \leqq 1$. The corresponding dependences are shown in Figs. 3 and 4.

The results of our analysis can be summarized in the following way, as can be seen from these plots. A nonoptimal orientation, $\vartheta = \pi/2$, lowers the amount of selected information, down to zero with $\chi = 1$, if there exists a nonzero contribution of the selective measurement, i.e., if $\chi > 0$. For $\chi > 0$, the dependence of the information amount on the entanglement parameter q is not very significant. The maximum of information, I_s = 1 bit, is achieved for the selectivity degree $\chi = 1$.

5. CONCLUSIONS

Compatible information is a unified characteristic of a quantum channel, which includes both purely classical and specifically quantum correlations in the spaces of input and output states. This concept is applicable to the description of information channels where the quantum entanglement of states does not change the structure of the space of states of an information system as a tensor product of the spaces of input and output states. The most general properties of compatible information described in this paper can be summarized in the following way.

The amount of unselected information, which is based on the consideration of all possible nonorthogonal input and output states, is independent of local transformations. However, selected information, which is based on incomplete sets of quantum states, substantially depends on local transformations at the input and output.

The reduction of POVMs corresponding to the replacement of unselected measurements by complete measurements leads to the increase in information amount only if the input and output are correlated in an appropriate way. This is due to the presence of an *a priori* information, which is, in fact, employed at the output if information selection is matched with the input and the state of the channel.

In the case of two qubits with no quantum entanglement and fully classical correlations relating the indices of orthogonal input and output states, the amount of unselected information is negligibly small, $I_u = 0.086$ bit. However, for a totally entangled input–output state, this amount is much larger, $I_u = 0.278$ bit, which characterizes also the information accessible in each of the qubits involved in information exchange. The 1-bit maximum of selected information and its zero minimum are achieved for a discrete orthoprojector measurement, with the maximum corresponding to a parallel orientation of quasispins related to input and output qubits and the minimum being achieved with a crossed $\pi/2$ geometry.

ACKNOWLEDGMENTS

This study was partially supported by the Russian Foundation for Basic Research (project no. 01-02-16311), “Fundamental Metrology” and “Nanotechnology” State Science and Technology Programs of Russian Federation, and INTAS (grant no. 00-479).

REFERENCES

1. Kholevo, A.S., 1973, *Probl. Peredachi Inf.*, **9**, no. 2, 31.
2. Hall, M.J.W., 1997, *Phys. Rev. A*, **55**, 100.
3. Grishanin, B.A., 1973, *Izv. Akad. Nauk SSSR, Tekh. Kibern.*, **11**, no. 5, 127.
4. Wald, A., 1950, *Statistical Decision Functions* (New York: Wiley).
5. Preskill, J., *Lecture Notes on Physics 229: Quantum Information and Computation*, http://www.theory.caltech.edu/people/preskill/ph229/.
6. Gallagher, R.G., 1968, *Information Theory and Reliable Communication* (New York: Wiley).
7. Schumacher, B. and Nielsen, M.A., 1996, *Phys. Rev. A*, **54**, 2629.
8. Lloyd, S., 1997, *Phys. Rev. A*, **55**, 1613.
9. Grishanin, B.A. and Zadkov, V.N., 2000, *Phys. Rev. A*, **62**, 032303.
10. Wooters, W.K. and Zurek, W.H., 1982, *Nature*, **299**, 802.
11. Dieks, H., 1982, *Phys. Lett. A*, **92**, 271.
12. King, C. and Ruskai, M.B., 2000, LANL e-print quant-ph/0004062.
13. Caves, C.M. and Fuchs, C.M., 1996, LANL e-print quant-ph/9601025.
14. Brukner, C. and Zeilinger, A., 1999, *Phys. Rev. Lett.*, **83**, 3354.
15. Lax, M., 1968, *Fluctuation and Coherence Phenomena in Classical and Quantum Physics* (New York: Gordon and Breach).
16. Grishanin, B.A., 1979, *Kvantovaya Elektron.*, **9**, 827.
17. Grishanin, B.A., 2000, *Quantum Random Processes*, http://comsim1.phys.msu.su/people/grishanin/index.html.
18. Grishanin, B.A. and Zadkov, V.N., 2000, *Laser Phys.*, **10**, 1280.
19. Bargatin, I.V., Grishanin, B.A., and Zadkov, V.N., 2000, *Phys. Rev. A*, **61**, 052305.
20. Barnum, H., Schumacher, B.W., and Nielsen, M.A., 1998, *Phys. Rev. A*, **57**, 4153.
21. 2000, *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation*, Bouwmeester, D., Ekert, A., and Zeilinger, A., Eds. (Berlin: Springer).
22. Bargatin, I.V., Grishanin, B.A., and Zadkov, V.N., 2001, *Usp. Fiz. Nauk*, **171**, 625.
23. Schumacher, B., 1990, *Complexity, Entropy and the Physics of Information*, Zurek, W.H., Ed. (Redwood City, CA: Addison-Wesley), p. 29.